On the Ancient Greek Mathematicians and their Mathematics

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Figure 1. Archimedes of Syracuse (c. 287 – 212 BC): his work represents the pinnacle of Ancient Greek Mathematics. Given the magnificence of their internal structure, Archimedes was in effect a composer of symphonies in Mathematics and Physics, not simply the author of treatises in these two fields. Together with Eudoxus of Cnidus, Archimedes created the field known today as Mathematical Physics. His alleged assassination by a Roman soldier signaled the beginning of the field's decline in the Helladic Space. Source of figure and credit to ref. [F1].

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Figure 2. Euclid of Alexandria (circa 325 – 265 BC). His ***Elements*** has been the most influential book in History, with the number of printings only second to the Bible. There are two mainstay methods found in Ancient Greek Mathematics, the Theory of Proportions and the Theory of Exhaustion; both were not initiated by Euclid, and were not perfected by him either. But they are the cornerstones of Euclidean Geometry, the Geometry that ruled the world of Mathematics till the 19th century, and they are found in abundance in Euclid's ***Elements***. Source of figure: ref. [F2].

Abstract

This paper is only a part of a larger research on, study and analysis of, Ancient Greek Mathematics and Astronomy by this author. It is intended for the general public. The mathematicians' **spatial and temporal density** within the Ancient Helladic Space over a millennium (c. 650 BC – 350 AD) is reviewed. The topic of spatial-temporal diffusion of that mathematical and astronomical knowledge is traced and discussed. A List of 55 Ancient Greek mathematicians and astronomers is drafted, schematically and hierarchically clustered around four (idiosyncratic of course, fuzzy and fluid) sets of major to minor Mathematics figures. The derived four-tier hierarchical structure contains individuals with a current recognition factor ranging from household names to obscure. Their brief Demographics are shown in a table, demonstrating the close correlation their spatial-temporal density had to the rise and fall of Classical Greece. It is argued that all these mathematicians and astronomers on the List (and numerous others not on the List, as History has not supplied enough information in the form of well documented written evidence on them to be included) formed the necessary infrastructure for, and played a significant role in, the development of what we currently view as the fields of Modern Mathematics and Astronomy. In addition, certain epistemological aspects of their work are looked at, in some detail and from a contemporary perspective. Fundamental matters of Epistemology they raised contain everlasting themes that resonate to this day. In this contemporary Era of intellectual deconstruction, the paper does not refrain from or mince superlatives to describe the real excellence of their work.

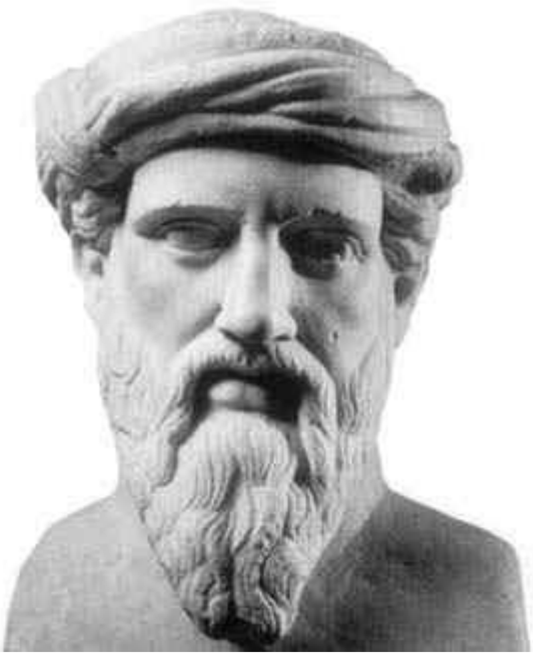

Figure 3. **Pythagoras of Samos (circa 570 – 490 BC)**. The 6th century BC **Pythagorean Theorem** set the stage for providing proofs to Propositions and established the process for mathematical reasoning and deduction. Pythagoras' School and the Pythagoreans formally established the Theory of Proportions, which was first put forward by Thales of Miletus, and proved to be of fundamental import within the discipline of Modern Mathematics. Source of figure: ref. [F3].

Prologue

This paper is part of a broader study, see ref. [A1], undertaken and underway at present by this author. Two related, off-shoots of this study, papers are found in refs. [A2] and [A3]. The available written record, of course unavoidably imperfect and incomplete, indicates that the Ancient Greek mathematicians dealt with complex and difficult subjects that were untouched till then. Irrationals, incommensurables, and arbitrarily close approximations to them were certain nodal topics they tackled, as they flirted with a host of concepts that today are referred to as “Calculus”, including those of zero, infinity and limits. Among numerous associated topics, they attempted to handle the Laws of Proportionality, and intersections of solids as well as adding time and motion to static till then geometric figures. These notions constituted some of their key, central, core concepts and brought up basic subjects in Epistemology. At the pinnacle of their creativity, Archimedes tried to confront head on key mathematical and epistemological issues, such as the challenges that approximations, limits and large numbers pose, all pivotal notions and still present in modern-day Mathematics, Astronomy, Geography and other related disciplines.

Ancient Greek mathematicians meticulously avoided using concepts that currently are used in abundance, notions such as zero, infinity, limits, negative numbers. They thought that these notions are not, and do not represent, Reality in their view of Nature, Society and Reality. Yet, the paradox is that they tried to (indirectly) argue against these notions and their (contemporary) meaning(s) by employing Geometry, geometric shapes and Algebra (as well as Trigonometry), i.e., means, modes and models that do not exactly replicate Nature and Reality either.

At the same time, they also ventured at the very core of matters of Logic, when they attempted to construct a theoretical model of Geometry that was based on an axiomatic structure, being consistent with some internal Logic founded on axioms, theorems, lemmas, and proofs; and they struggled to some degree with the inner workings of that structure, its implied contradictions in both Mathematics and Logic. They did not get to the role the subject of Linguistics plays, a topic that had to wait till the 19th century to be appreciated, as it intrinsically relates to these fundamental concepts. They did not also realize the foundational role that the concepts of **scale** and detail hold in the worlds of Reality and Virtual Reality, i.e., the mathematical models which observers of Nature concoct to analyze and study Nature. The importance of scale and implicit **resolution**, although raised by them, must wait till the 21st century to be fully recognized in Epistemology. But their struggles with such fundamental notions and perceptions resonate to this day. This is the angle to be taken, from the New Epistemology viewpoint, that makes worth revisiting the Ancient Greek mathematicians and their very innovative for that Era Mathematics.

A List of 55 Ancient Greek mathematicians (which includes also astronomers and to some limited extent philosophers, who were involved with Mathematics) is drafted. The List constitutes a key part of this paper, as its drafting brings about numerous issues that relate to the epistemological foundation of their work. Their extraordinary spatial-temporal density is also explored in **Note 1**, which also addresses the issue of population size during the period, a factor that must have contributed to their 8th century BC explosion in colonies throughout the Mediterranean Basin.

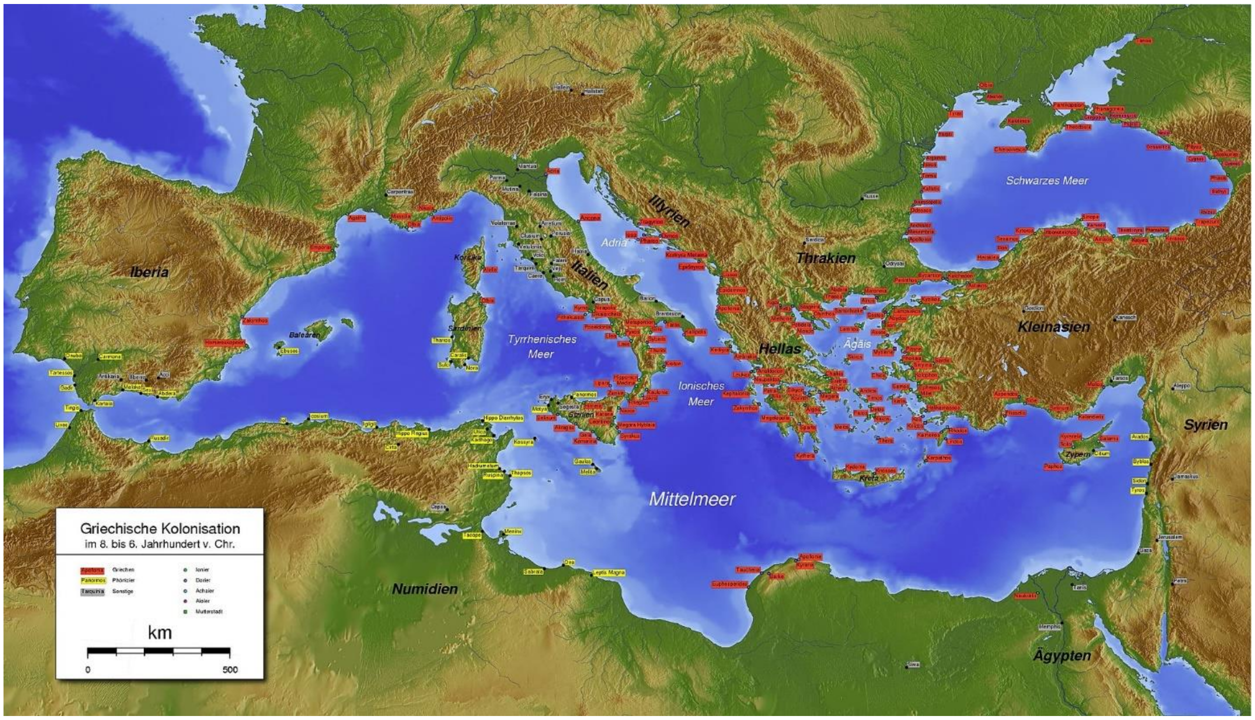

Figure 4. **The Helladic Space of the circa 700 BC period.** In red, the Ancient Greek mainland cities, settlements and colonies in the Mediterranean Basin and the Black Sea, while in yellow, the so-called "Phoenician" colonies are shown. Apparently, a significant Ancient Greek **out-migration**, under push-pull factors at work and possibly under explosive endogenous **population growth**, occurred at around 750 BC. See **Note 1** at the end of this paper on this aspect of the above map. It is also rather certain that the Ancient Greeks strongly interacted in a two-way flow of influences, with the Egyptians and Babylonians prior to, during, and at the aftermath of the so-called **Greek Dark Age** that followed the collapse of the Minoan and Mycenaean civilizations. Spatial-temporal dynamics of these flows and interactions are rather difficult to trace in detail, due to lack of documented evidence. Some guidelines on how to look at them are suggested here and in ref. [A1]. The end of the Greek Dark Age saw the emergence of **History**, formal rather than anecdotal and informal (as either oral tradition or lithic inscriptions), albeit still fuzzy storytelling. This History contains an increasingly reliable, subject to cross-reference, written record that can be construed as adequate evidentiary material on the basis of which the arguments of this paper are structured. Influences prior to the Greek Dark Age among the Asia Minor, Mainland Greece, and the Apennine Peninsula and its maritime environs (**Notes 1, 2**) can be traced on extant (albeit transformed in numerous ways) structures and artifacts, see ref. [A4]. A number of factors which played a role in the evolution of Mathematics and Astronomy in this Region (shown in the above map) are discussed in ref. [A1]. For example, the Arithmetic system used throughout the Upper Eastern Mediterranean Basin during the Late Bronze and Iron Ages among the Minoans, Mycenaeans and the Hittites, was apparently similar and decimal based, see ref. [A2]. On the architectonic influences among the Helladic Space, Hittites and Sardinia's Nuraghe cultures during the 2nd millennium BC see ref. [A4]. Source of the above map and credit to: ref. [F4].

Introduction

After more than two millennia that have elapsed, after more than a century of intensive scrutiny upon the Ancient Greek Mathematics and Astronomy following the seminal work of the late 19th and early 20th century mainly (albeit by no means exclusively) by Thomas Little Heath (see refs. [1] – [6]), one may wonder if there is much left to be said about that pioneering work that occurred over a chronological time span extending roughly from the 8th century BC (i.e., from what is referred to as "Archaic Greece"), through Classical Greece and the Hellenistic period, to the 3rd century AD (and the pre-Byzantine Era), work that took place in the Helladic Space, a rather wide geographic area that approximately included continental Greece, the Aegean islands, Ionia at the Western Coastal region of Asia Minor, Magna Grecia at the Apennine Peninsula and Sicily, the Seleucid Western Asia, and Ptolemaic Lower Egypt's Mediterranean Coast, among various domains, see Figure 4.

It is the firm conviction of this author that much still is, and has yet, to be said about that extraordinary and admirable assembly of bright minds that we currently refer to as "the Ancient Greek mathematicians and astronomers" and their Mathematics and Astronomy. And what can be said, from a vantage point of two millennia later, might be as important as what they envisioned their work to be back then. For what they did and what they said still resonates across the contemporary world of Mathematics and Astronomy, Philosophy, Logic, Linguistics, and basically Epistemology. That two-millennia long resonance has amplified some of their sounds, sights and meanings, and has produced echoes that even the Ancient Greek mathematicians and astronomers had little imagined then that it would last for so long and would still be relevant.

It is that echo which this paper attempts to capture and convey to the modern-day reader.

On-going work on this paper had a humble beginning; it basically started as an attempt to revisit, analyze and scrutinize Eratosthenes' celebrated and widely discussed experiment to measure the Earth's polar circumference, c. 240 BC, while he was the head Librarian at Alexandria's Ptolemaic Library, see ref. [A1]. The study's main focus is to demonstrate that the Eratosthenes work was carried out within a long in time and widespread in space intellectual environment and tradition, an intellectual milieu which sustained that work. Eratosthenes experiment was the product of a rich and sophisticated intellectual morae in both Mathematics (particularly Geometry) and Astronomy. It is within this scope, framework and context, that this paper is written.

The seminal work by Thomas Little Heath on the Ancient Greek Mathematics and Astronomy, in part summarized with ref. [3], on the one hand documents the strong evolutionary linkages among the extraordinary intellectual achievements of great Ancient Greek mathematicians and astronomers; but on the other hand, it also points out both the fuzziness, fluidity and boundless richness of their indeed amazing, both dazzling and profound (even by current mathematical standards, but still enormously impressive and influential well over 25 centuries later) discoveries. Heath's work stimulates the modern-day reader into formulating novel hypotheses on topics that the Ancient Greek Mathematics and Astronomy ventured into addressing.

Glittering geometric gems still sparkle and shine brightly within the Literature on Ancient Greek Mathematics. They include, without being limited to, topics such as: the analytical (algebraic and geometric) components of a quadratrix, conchoids, cissoids, the Theodorus and Archimedean spirals, the treatment of irrationals and incommensurables. On these subjects, much is till to be said, even beyond the mathematical detail that has been produced since their re-discovery during the Renaissance to the present time's Analytical Geometry and Advanced Mathematics.

These are topics which entail complex Geometries that evolved in the Greek mathematicians' minds out of trying to innovatively approach standing (some long before them) mathematical puzzles, such as doubling the cube (the Delian problem) and squaring the circle, trisect any angle, find the quadrature of lunes, study polyhedrons like icosahedra (ΕΙΚΟΣΑΕΔΡΑ), etc. All that while deriving a systematic theory of numbers by setting the exact mathematical *formulae* and processes to generate Primitive Pythagorean Triples, natural and triangular numbers, perfect and friendly numbers, very large numbers, arithmetic and geometric series and complex 2-d re-arrangement games called Ostomachia (ΟΣΤΟΜΑΧΙΑ). How these problems were conceived, approached and solved is the subject matter of ref. [A1], which also focuses on the works of certain key mathematicians and astronomers (Thales, Pythagoras, Eudoxus, Callippus, Euclid, Aristarchus, Archimedes, Apollonius, Hipparchus, Pappus being some of them), not all household names but central in the development and evolution of Ancient Mathematics and Astronomy. This study also attempts to debunk modern-day claims that challenge (and even deny) the very existence of key figures among them, such as Pythagoras and Euclid, by modern-day deniers.

Ancient Greek mathematicians devised the Theory of Mean Proportions, the Golden Ratio and the Theory of Exhaustion in solving mathematical problems through pure Geometry. By the use of simply a ruler and a compass, they were forced to deal with Geometries involving finding the roots of second, third and even higher degree equations, trace the locus of points forming complex nonlinearities in curves and surfaces. All of that without Algebra, although they were able to geometrically obtain the positive roots of quadratic and in certain limited instances of cubic equations. They formally stated the complex forms and Mathematics of geometric shapes (and their intersections) such as ellipses, parabolas and hyperbolas, cones, cylinders, pyramids, spheres and tori. Key features among these topics are extensively reviewed in ref. [A1]. Here the emphasis is on the epistemological aspects of these topics. The Ancient Greek mathematicians ingeniously dealt with subjects that modern-day Integral and Differential Calculus addresses, but at an even more epistemologically fundamental manner, as it will be argued in this paper.

They introduced stunning geometric diagrams, by introducing **time** into the (till then) static part of Geometry. Initially through the pioneering work on irrational numbers and lengths by Eudoxus of Cnidus on the Theory of Proportions (or Mean Proportions) and his extraordinarily advanced for that Era work on retrograde movement (to be addresses later in some further detail); and with the discovery by Theodorus of Cyrene of the spiral that bears his name (also to be discussed more at length in this subsection); that body of work set the stage for deriving a Dynamical Geometry with moving individual parts and overall perspective (see Conclusions on this topic).

That work reached its peak with the work by Archimedes, some of it extant in some form, including his recently discovered Palimpsest, work that will be briefly reviewed in a subsequent section of this paper. These mathematicians' advances and discoveries are more extensively covered in ref. [A1]. Ancient Greek architects and artisans, inspired by the stellar findings of their mathematicians and astronomers' work and accomplishments, implanted their mathematical, geometric and astronomical findings into their monuments and artifacts, so that influences by specific mathematicians and astronomers on them can be traced today, as modern Mathematics can be traced on contemporary Architecture and Engineering. More on this topic shortly.

Scholarship, peer review and publishing

Mathematicians of that Era, in cooperation with secular academic institutions that were formed for the first time (not only in the Helladic Space but anywhere History covered), set up a system of reviewing and publishing their work, a system which has led to the modern-day scientific peer review process. It is remarkable that this rigor in scholarship can be documented only to some extent by what these thinkers wrote, since in most cases their work has been lost, but mostly by what others said about them and their mathematical and astronomical work after reviewing it.

Peer review and publication of mathematical and astronomical work took place at the major centers of learning and depositories of knowledge that were established back then. In support of that proposition, there are well documented written records. Two of these cases will be discussed here, indicative of the depth information dissemination scholarship, and specifically Mathematics related work was subjected back then. One is related to Plato's Academy, while the other to the Library of Alexandria. Both are connected to individuals that were among the brightest on that List of 55: Plato and Archimedes. In his seminal work *Theaetetus*, ΘΕΑΙΤΗΤΟΣ, c. 369 BC, Plato presents (for the first time in the History of Philosophy, Mathematics and Astronomy) dialogues among scholars, and in effect formal epistemological discourse taking place in his Academy about the nature of Knowledge in general, Mathematics in specific, and more precisely about irrationals and quantities that were thought then, and are referred to today, as "incommensurables". Theodorus (ΘΕΟΔΩΡΟΣ of Cyrene), and reportedly teacher of Plato in Mathematics, is mentioned as: "…proving to us certain things about square roots…", see ref. [1], p. 203. The term "proving to us" in this sentence points to the element of scholarship. Thomas L Heath, argues with prior translators of Plato's *Theaetetus,* who translated the Greek text's term ΕΓΡΑΦΕ (a term which literally means "wrote") whether the meaning of the term should be taken to be "drew" or "constructed". The embedded substantive issue needs further exploration.

The argument is of mathematical consequence, beyond its linguistic and literary angles. It entails the epistemological aspect of the fact that Theodorus dealt with square root of integers from 3 to 17, skipping that of 2 and not continuing past 17. The prevailing presumption is that Theodorus took it as given that Pythagoras had already addressed the irrationality of 2^1/2 prior to him. However, the issue remains still open as to why Theodorus' (implicit) argument that, with number 17 the first round (revolution) of the "Theodorus spiral" is completed, was not debated in the Academy and did not preoccupy Plato and his colleagues any further.

“Theodorus spiral”, a spiral defined by discrete-step approximations to a continuous flowing ΕΛΙΞ, has a structure that commences with an isosceles right-angled triangle formed by legs equal in length to a unit (1), and where each successive right-angled triangle has as its long leg the size of the prior triangle’s hypotenuse, and as a short leg the unit length. The hypotenuses hence are: 3^12, 4^1/2, 5^1/2, etc., with the last, the 16th, triangle being the one with hypotenuse 17^1/2. The impressive, innovative in many respects, “Theodorus spiral” is discussed at length in ref. [A1]. What becomes rather clear from the above reference to Plato’s *Theaetetus* is the continuity, rigor (for the time) and scholarship involved in the reviewing process implied by the reference.

The second stellar example of scholarship, peer review and existence of publication records involves Archimedes. Numerous extant works (classified as “Books”) of Archimedes, see the volume by T. L. Heath in refs. [5] and [5.1], contain Prefaces at their beginning. These Prefaces take the form of a forwarding letter (epistle, ΕΠΙΣΤΟΛΗ) and act as an “Introduction to the Book” document. In that introductory letter, Archimedes briefly describes the work involved and he explicitly mentions his intention to have that work “published”. All his works he intended to “publish”, have them read by “knowledgeable scholars and geometers” (all being terms that he explicitly uses in these Prefaces), eliciting responses and have them disseminated by the intended receiver of his Books. That receiver was apparently the contemporaneous Head Librarian of the Great Library at Alexandria. One of the intended recipients, specifically of his recently discovered (in 1906) volume (the so-called “Archimedes Palimpsest”), which contains his Book on *The Method*, was Eratosthenes, see ref. [5.1], p. 12. Five of his Books (see ref. [5], on: *The Sphere and Cylinder, Books I and II; Conoids and Spheroids; Spirals; Quadrature of the Parabola*) were sent to someone by the name of Dositheus (likely at the Library of Alexandria). Of unknown background, Dositheus was probably another Head Librarian, or Assistant to the Head Librarian, or some high-ranking public official in that Great Library, for Archimedes to entrust him with his manuscripts.

These two items, one related to Plato’s open and public treatment of mathematical material presented by Theodorus; and the other related to Archimedes’ desire to have his work presented to colleagues at the Library of Alexandria, constitute strong evidentiary material to support the proposition that scholarship and a reviewing process were present back then, and were as essential as in modern-day scholarship. The reviewing process by peers and the existence of outlets for “publication” were functions apparently being performed back then by institutions such as Plato’s Academy, Zenon’s Stoa Poikile Academy, and the Great Library at Alexandria.

In these “introductions” by Archimedes to his Books, as expressed in the epistles addressed to the various intended receivers of his work either soliciting comments from “knowledgeable individuals”, and/or asking for publication of his work, certain weaknesses of western scholarship are also detectable. In some of these letters, Archimedes anonymously refers to “other previous geometers” whose work he indirectly (albeit politely) criticizes; and he also refers to reviewers of his own work, who (according to Archimedes) have failed to fully understand and appreciate his Propositions’ deeper meanings. Moreover, his work was sent to individuals he apparently had either personally met or knew well, and with whom he had established a personal friendship.

The fallout of the Greek Dark Age

Ancient Greek Mathematics evolved to some degree **endogenously** within the Helladic Space, as well as they were **exogenously** influenced by sources outside that loosely defined domain, such as Mesopotamia and Egypt, albeit not exclusively so. The map of Figure 4 is quite informative on this count, as it shows (even from early on, circa 700 BC) an extensive flow, spreading and diffusion of cultural influences within and around the Mediterranean Basin by Greek colonists. Hence, ideas connected to Mathematics and Astronomy must have been disseminated to these settings from the Helladic Space as well. There is absolutely no reason to believe that these flows were only one-way transmissions and transactions from outside the Helladic Space into it. Evidence, at least, from abundant shipwrecks laying on the Mediterranean seabed is obviously indicative of a strong intertemporal interaction and diffusion of commodities, populations and ideas, among spatial settings dispersed within the Basin, and beyond its geographically, climatologically and (now and then) aesthetically appealing seashores. More on this evidence is presented in ref. [A1].

Influences upon Ancient Greek mathematicians and astronomers emanating from Dynastic Egypt and Mesopotamia (and, more broadly, from the Levant) have been anecdotally reported; see ref. [A1] for more detail on such evidence. Perceptions of mathematical objects (triangles, rectangles and circles); and the plethora of records kept by both Egyptians and Babylonians regarding a considerable variety of astronomical events consisted very likely a body of knowledge and stock of data with which, to a large extent, Helladic Space’s scholars must have been familiar.

However, it is now evident that these objects and events were approached by both Egyptians and Babylonians as topics obeying certain **empirical regularities**. They were simply the subjects of intellectual curiosity, elementary aesthetic appeal and practical architectonic and engineering applications. Record keeping in both Mesopotamia and Egypt must have predated the rise of analysis and theory formation in both Mathematics and Astronomy as experienced post 700 BC within the Helladic Space. Many of these regularities in both geometrical shapes and astronomical events, looked at as material curiosities of intellectual interest by both Egyptians and Babylonians, must have been imported into the Helladic Space from outside it. Especially so prior to and during the Greek Dark Age, on which see **Note 2**, a period stretching approximately from the currently perceived collapse of the Minoan and Mycenaean Civilizations to the Archaic Period, a temporally fuzzy period within the time frame of circa 1100 to 1050 BC – 800 to 750 BC.

The topic of intellectual influences in the pre-Greek Dark Age period is far broader than simply the potential flow of influence from the Egyptian and Mesopotamian sources into the Helladic Space. That complex and largely unrecorded flow of influences must have extended within the two nodes (Egypt and Mesopotamia) as well; and also, to have included other key sinks and sources, such as Asia Minor and the Hittites, the Upper Balkan Peninsula, and the Middle Mediterranean settings of Malta, Sardinia, Sicily, and the Lower Apennine Peninsula. It was a complicated web of interactions, that pre-History has been quite negligent in recording; even a *modicum* of evidence is lacking in that regard, see ref. [A1] on this interconnectivity subject.

Following this obscure period of upheaval, the period encompassing the end of the Iron Age, and the flow of the Dorian tribes from the North (the Balkan Peninsula and possibly even North of it) into the Helladic Space, a new Era begins for that region and central geographic focus of this paper: **Recorded History**. The transitional border, the 9th and 8th centuries BC, is a relatively period incompletely reported by History to a very large extent and only approachable in a fuzzy manner today. It was the period that the Homeric Epics and the Hesiod Poetry appeared in writing, metamorphosing (with leaps and bounds) an oral to a written literary tradition (**Note 2**).

A new Era in Mathematics and Astronomy dawned then and there; it brought with it the belief that **propositions** and **generalizations** about mainly mathematical (geometric and astronomical) objects can be formed and stated in terms of formal Theorems and Propositions and Lemmas; and above all, that these propositions could be systematically and methodically proven; in turn, that these **proofs** could be self-consistent and always based on strict definition of terms and the establishment of **axioms**. And that these axioms must be congruent with some internally consistent **Logic**. Thus, the Ancient Greek mathematicians and astronomers set the stage for a rigorous approach to Logic and an axiomatic statement of all mathematical thought and statements. They hence set up Mathematics as we know the field today.

In summary, **something very dramatic** must have happened during that time period that is referred to as "The Greek Dark Age", which by c. 800 – 750 BC triggered a deeply felt change. The Helladic Space entered a spectacular new dynamical path, and local as well as neighboring regions' population and per capita dynamics must have played a significant role in it. On the subject of population stock and changes see **Note 1**, and on the subject of population-income dynamics, see refs. [A8], [A9]. These new population and income dynamics re-shaped the what is known to contemporary Western thinkers as Western Civilization and its History. Unfortunately (or maybe fortunately, since no full assessment, knowledge, comprehension and appreciation of the pleasant and unpleasant social-spatial-temporal dynamics that occurred then is at present possible), History is still very obscure as to what this event (or, most likely, combination of events) must have been to bring about such scientific, mathematical and intellectual revolution. Undoubtedly, large in scale (peaceful as well as hostile) population in- and out-migration dynamics, ecological changes, technological innovations in both production and consumption as well as in transport modes, shifts in demands for natural resources, natural events, etc., must have contributed (in both positive and negative ways) to such a dramatic shift. Switches in dynamics are extremely difficult to detect in both their (usually multiple) causes and effects, especially under conditions where concrete evidence is lacking. Multiplicity of interpretations of whatever empirical evidence can be collected and reviewed compounds those difficulties.

No matter what must have triggered the transition from the Greek Dark Age to the Archaic Greek world, it was an intellectual earthshaking revolution. It marked the passing of a **threshold** involving a critical **transition**, within the ensued century: that (fuzzy) threshold commenced roughly around middle 8th century BC, the time of the first Olympiad and the historically recorded first Ancient Greek personalities, possibly Homer and far more likely Hesiod; see also **Note 2**.

It occurred while a strong Greek out-migration flow marked the establishment of Greek colonies around the Mediterranean Basin, as shown in Figure 4. The threshold lasted till the middle of the 7th century BC, and the formation of the Miletus School by Thales and Anaximander. The passing of that threshold saw the transition from a phase of observing and recording empirical regularities to the formulation of scientific hypotheses and the establishment of mathematical proofs. The Earth was no longer flat, and the right-angled triangle (3,4,5) had a profound story written all over and inside it to tell. This is what the Helladic Space produced and partially exported to the neighboring region(s) over the ensued millennium (roughly from 650 BC to 350 AD); and all that intellectual explosion happened following that obscure period called the "Greek Dark Age", which saw also the introduction of the Greek alphabet in the region, see **Note 2**.

Figure 5. Eudoxus of Cnidus (circa 408 – 355 BC). He introduced the **Theory of Exhaustion** in Geometry, and he also added **temporal motion** to geometric schemata. Through the use of Dynamical Geometry, involving rotating concentric spheres, he was able to produce retrograde apparent movement in an effort to model the retrograde motion of planets in the night's sky. His dynamics did not work well for Astronomy; but that thought was a revolution for Mathematics. Eudoxus influenced to a significant degree both Euclid and Archimedes' works. He may not be a contemporary household name, but he was a genius. Source of figure and credit to: ref. [F5].

Eponymia and accomplishments in the world of Ancient Greek Mathematics

On her classical work regarding the Babylonian tablet referred to as Plimpton 322, Eleanor Robson, see ref. [7], p. 117, mentions that "Ancient Mesopotamia was a culture that prized anonymized tradition over individual creativity". Be that as it may, and to the contrary, Ancient Greek culture emphasized and was built around the importance of the individual. Individual contribution, creativity and innovation, i.e., Eponymia rather than anonymity, as well as individual distinction, welfare and wellbeing were prized qualities in Ancient Greece.

One may venture to say and rather safely speculate, even presume, that this was the founding element to, and the cornerstone of, the Ancient Greek Civilization and the driving element towards a societal structure, the very notion of a ΠΟΛΙΣ. Eponymia and knowing who created what and when, were the lever that lifted the Humanities, Arts and Sciences to heights that were remarkable for that Era, and not only for the World of the Mediterranean Basin, but Worldwide.

Within a span of approximately a millennium, from circa 650 BC to about 350 AD, a remarkable number of individuals were preoccupied with what we currently consider to be Mathematics, Astronomy and Geography, the building blocks of all the Arts and Sciences, the Humanities and many Professions. These individuals rose to prominence, and they left a deep and lasting mark into the World's knowledge base, Civilization and Humanism.

Many among them transcended the old religious molds, legacies and intellectual constraints of the Paleolithic, Neolithic and the Metals Ages, as they approached the world of intangible, abstract, intellectual ideas. They rose above the mandane and ordinary, conventional (apparently largely and often miserable, till that point in space-time) individual human life, experience and existence. It must be said, and it is well documented (see ref. [7]), the vast majority of individuals who were mathematicians and astronomers, at the time period under review, did belong to the economic and social elites of their time. They had the means to travel widely, and they were well educated (relative to that Era's prevailing on the average level of education); they afforded to attend the top educational institutions of the time, and even benefit from the privilege of having private tutors. They circulated among the upper echelons of the local social and political elites, some of them becoming distinguished members of those socio-political elites as well.

A more detailed account of certain among these individuals' lives (for example, the Miletus' Thales and Anaximander, Pythagoras of Samos, Eudoxus of Cnidus, Callippus of Cyzicus, Euclid of Alexandria, Archimedes of Syracuse, Apollonius of Perga, Pappus of Alexandria) is offered in ref. [A1], where the life and work of Eratosthenes is also included (the subject that triggered this author's more extensive study of the Ancient Greek Mathematics and Astronomy).

In turn, these mathematicians and astronomers fed that upper social stratum with their knowledge and intellectuality. The underwriters of the Greek Golden Age, those who provided the intellectual infrastructure upon which the culture of Classical Greece was built, were the Ancient Greek mathematicians, and especially their geometers. Greatness in Ancient Greek Mathematics was the leading indicator to greatness of the Ancient Greek Civilization. And, in a paradoxical feedback loop, that greatness of Civilization was a leading indicator into the ensued decline of the mathematical field. It was a Civilization that entered a phase of further decline when its Mathematics and Astronomy waned. Classical Greece and the Hellenistic period were founded on their Mathematics and Astronomy. In a ripple and cascading effect, a number of fields directly benefited from that greatness, from Geography to Architecture, Engineering and the Arts. Advances in Mathematics and Astronomy were imprinted on the design of their monumental structures. Only a few will be very briefly listed below.

The 5th century BC (second and possibly stratigraphically third and final) Temple of the Parthenon at the Acropolis is a stellar example of templar Architecture, where the architects Ictinus (ΙΚΤΙΝΟΣ) and Callikrates (ΚΑΛΛΙΚΡΑΤΗΣ) applied detailed and careful approximations to irrational numbers (specifically the Golder Ratio, **φ** = [1 + 5^1/2]/2 ≈ 1.61803…) to the structure's facades, see ref. [A10]. Another example is Ictinus' floor plan design of the late 5th century BC Temple of Epicourius Apollo (ΕΠΙΚΟΥΡΙΟΣ ΑΠΟΛΛΩΝ) at Bessae (ΒΑΣΣΑΙ), where an approximation to the Silver Ratio (**δ** = 1 + 2^1/2 ≈ 2.41421…) has been imprinted, see ref. [A11].

On the Kasta Tumulus at Amphipolis, a late 4th century BC structure, a very close approximation to the Earth's elliptical orbit around the Sun has been implanted. The tumulus site plan design, possibly drawn up by Alexander III imperial architect Deinokratis (or maybe Parmenion, or both, as both were then involved in the design of Alexandria's site plan as well as various public buildings there), apparently was influenced by contemporaneous mathematician and astronomer Callippus of Cyzicus (c. 370 – 310 BC), in turn influenced by Eudoxus. In the modular structure of the (incomplete) artificial and meticulously drawn ashlar, marble-clad wall that surrounded the tumulus, i.e., on the circumference of that monument, space was made available to accommodate 365.25 marble stones, each of a current length (measured in modern-day units) approximately 1.60 cm; on the monument's Architecture and Astronomy, see refs. [A12], [A13], [A14]. The examples mentioned are just a few of the structures where the Ancient Greek Mathematics and Astronomy have left their mark. Architecture is just one field where these marks are detected. Technology and Engineering, the Arts and Sciences (with artifacts such as the Antikythera Mechanism and various sculptures, including three masterpieces, the c. 450 BC Myron's ΔΙΣΚΟΒΟΛΟΣ, c. 150 BC Alexander of Antioch's ΑΦΡΟΔΙΤΗ ΤΗΣ ΜΗΛΟΥ, and c. 2nd century BC ΝΙΚΗ ΤΗΣ ΣΑΜΟΘΡΑΚΗΣ, see ref. [A15]) were all heavily influenced by the pioneering work and innovative thinking of the mathematicians that are on the List that follows.

It is ironic but certainly not random that Archimedes, possibly the greatest mind of Antiquity, was allegedly killed by a Roman soldier who apparently disobeyed his commander's order that his life be spared during the seize of Syracuse in 212 BC. At the aftermath of these foreign conquests, that land of the Helladic Space remained largely aloof, impoverished, devastated, ravaged from pillage and lamenting ever since. In that light, it is illuminating to review the impressive List of those who left their individual marks and heavy fingerprints on that History of Ancient Greek Mathematics and Astronomy, with their names imprinted on the brightly lit Wall at the Pantheon of intellectual accomplishments in Mathematics and Astronomy, from that millennium-long time period. The reader should recognize that the actual List of mathematicians and astronomers of that Era very likely contained a far greater number of individuals than the 55 that are mentioned here, as History has certainly not retained all of their names, and not fully credited all. In fact, it is safe to presume so, as only those who made extraordinary contributions and had the lack of the draw had most likely their name recorded in History's books. Nonetheless, even with this *caveat* in mind, the scale of that magnitude and fortitude of intellectual power is impressive. The diversity of thought they garnered became admirable, astonishing and awe inspiring to those who followed, and remains so still to the present day.

It imposed however a heavy burden and legacy to carry for those that succeeded them, in a severely reduced and battered by foreign (Roman and other) conquest and subsequent invasions and pillage of that land. In spite of all that misfortune, the light they shed on Mathematics and Astronomy still shines bright, more than two millennia later.

The List of 55 Ancient Greek mathematicians and astronomers

Introductory comments

There is a relatively well recorded, written and rather adequately, albeit imperfectly and of course incompletely, documented as well as uninterrupted (with overlapping life spans) chain of mathematicians and astronomers that can be traced *ante* and *post* Eratosthenes, who all lived within the Helladic Space, a fuzzy region which (as shown earlier) extended from South-Eastern Europe into Western Asia and Northern Africa. They all to a large extent communicated, critiqued, referenced and knew (at least something about) each other.

Below, italicized bold characters designate current household names of noted and distinguished Ancient Greek mathematicians. Some of them are primarily viewed as either astronomers or philosophers, emphasizing the **fuzziness** in labeling individuals' professions, a cornerstone of the New Epistemology and Fuzzy Sets Theory. In total four (fuzzy) groupings are shown progressively identifying major and minor figures in the World of Ancient Greek Mathematics: regular, lowercase typing font (37); lowercase, italicized font (5); lowercase, italicized, bold font (6); and uppercase, italicized, bold font (6). This rough, fuzzy, fluid, hierarchical classification is based on the impact these bright individuals had; it is largely a subjective judgement made by this author. Others would have made changes, either basic or at the margins, to that fuzzy classification. The name of Eratosthenes is in bold as it separates the *ante* and *post* period for the study in ref. [A1]. No matter how imperfect, incomplete and restricted any hierarchical decomposition scheme one could devise to classify and label these 55 individuals, it can be rather safely stated that no other socio-spatial setting in both the Region under review (Western Eurasia and North-Eastern Africa), or any other part of the world during the entire pre-historic and historic Antiquity came even close to the grandeur of these individuals' mathematical and astronomical achievements, all found within the spatial-temporal context of the Helladic Space. The extraordinarily high **spatial-temporal density** of these individuals, their consistent in spatial-temporal diffusion pattern over the space-time context of the millennium-long lasting Helladic Space, and thus the underlying **scale in relative population size** and their operation (recorded by the written reports on their life and work as mathematicians and astronomers) was unprecedented. On this angle of scale, see **Note 1** at the end of this paper. The full appreciation of that scale strikes the analyst when the then population size of the Region under study (the Mediterranean Basin and its immediate environs) as well as that of the Helladic Space are taken into account. Approximate and rough estimates of that population stock size currently available provide a perspective as to the scale of that activity, by today's standards.

In combination, these 55 individuals comprised the **necessary infrastructure** that fueled mathematical and astronomical developments in that millennium within that spatial domain of activity. They formed a spatial-temporal hierarchical dynamic network, of overlapping in space-time lives and influences. That dynamic network comprised of and involved direct communication among them. It was an inter-generational interaction. Each of these individuals was a node, around which a cluster of other supporting individuals (assistants and associates most of them lost to History) assembled. The hierarchical breakdown (6, 6, 5, 37) seems to suggest that History has failed to account for at least fifteen mathematicians (most of the lowercase italicized font type); and that Archimedes could be at a level above all (in a schema 1, 4, 8, 15, 40 for a total of about 70). These individuals and their mathematical work were not a flash in the pan type of activities, but a well-knit (albeit fuzzy) group of scholars acting as an assembly under a common objective, their shared goal being the advancement of mathematical and astronomical knowledge and understanding. Flows of ideas among them have been to an extent recorded, indicative of ideas at the stage of formation that evolved from streams into rivers of thought, in a sequential manner. In the List that follows, all of the dates offered are approximate (some maybe grossly so), and the places mentioned ought not to be taken literally, for reasons that are made clear in the text of ref. [A1], consistently with the basic cannons of the New Epistemology, as stated by this author in refs. [A5], [A6], [A7]. Numerous sources were consulted by this author in drafting up this list. Of course, classical sources such as Thomas L. Heath (ref. [1] – [6]) assumed a dominant role, and occupies a par excellence unique position in the drafting of the papers in refs. [A1], [A2], [A3] and in the List. However, auxiliary and secondary sources were used as well, as for example that in ref. [8]. Sources will be discussed in turn.

The List of 55

This remarkable chain, of 55 (there could be, and certainly were far more) mathematicians and astronomers, that lasted for about a millennium (from roughly 650 BC to about 350 AD, hence having one appearing every two decades on the average), commences with ***THALES*** (c. 624 – 547), and *Anaximander* (c. 611 – 546) both of Miletus, and the beginning of modern (proto) Ancient Greek Science and Mathematics fields, disciples which, from then on, were strictly "based on proofs"; Anaximenes of Miletus (c. 586 – 526 BC), ***PYTHAGORAS*** of Samos (c. 570 – 490), Xenophanes of Colophon (c. 570 – 478), *Heraclitus* (c. 535 – 475 BC), Parmenides of Elea (c. 515 – c. 440), Anaxagoras of Clazomenae (c. 499 – 428 BC), Empedocles of Akragas (c. 494 – 434), Oenopides of Chios (c. 490 – 420), Philolaus of Croton (c. 470 – 385 BC), *Democritus* of Abdera (c. 460 – 370 BC), Hippias of Elis (c. 460 – 400 BC), Hippocrates of Chios (c. 470 – 410 BC), Theodorus of Cyrene (c. 465 – 398 BC), Theaetetus of Athens (c. 417 – c. 369 BC), *Archytas* of Tarentum (c. 428 – 350 BC), ***PLATO*** (c. 427 – 347 BC), ***Eudoxus*** of Knidos (c. 408 – 355 BC), Xenocrates of Chalkedon (c. 396 – 314), Dinostratus of Alopeconnesus (c. 390 – 320), Heraclides of Pontus (c. 387 – 312 BC), Theudius of Magnesia (4th century BC), ***ARISTOTLE*** (c. 384 – 322 BC), Menaechmus of Alopeconnesus (brother of Dinostratus, c. 380 – 320 BC), Callippus of Cyzicus (c. 370 – 300), Aristaeus the Elder (c. 370 – 300 BC), Autolycus of Pitane (c. 360 – 290 BC), Eudemus of Rhodes (c. 350 – 290 BC), ***EUCLID*** of Alexandria (c. 325 – 265 BC), ***Aristarchus*** of Samos (c. 310 – 230 BC),

ARCHIMEDES of Syracuse (c. 287 – 212 BC), Nicomedes (c. 280 – 210 BC), Philon of Byzantium (c. 280 – 220 BC), Conon of Samos, (c. 280 – 220 BC), Pythias the Thasian (unknown thereabouts, 3rd century BC), **Eratosthenes of Cyrene and of Alexandria** (c. 276 – 194 BC), ***Apollonius*** of Perga (c. 262 – 190 BC), Dionysodorus of Kaunus (c. 250 – 190 BC), Diocles of Arcadia (c. 240 – 180 BC), Persus (unknown whereabouts, likely second century BC), Zenodorus of Athens (c. 200 – 140 BC), ***Hipparchus*** of Rhodes (c. 190 – 120 BC), Hypsicles of Alexandria (c. 190 – 120 BC), Theodosius of Bithynia (or Tripolis, c. 160 – 90 BC), Zeno of Sidon (c. 150 – 70 BC), Posidonius of Rhodes or Apameia (c. 135 – 51 BC), Geminus possibly of Rhodes (c. 10 BC – 60 AD), Heron of Alexandria (c. 10 – 75 AD), Menelaus of Alexandria (c. 70 – 130 AD), *Claudius Ptolemy* of Alexandria (c. 85 – 165 AD), ***Diophantus*** of Alexandria (c. 200 – 284 AD), Sporus of Nicea (c. 240 – 300 AD), ***Pappus*** of Alexandria (c. 290 – 350 AD).

A few comments on the List, sources used and the historical record

Numerous other individuals that could have been included on this List are not, for a host or reasons. For example, on some of those missing from the List there exist considerable doubts as to their identity. This is the case for instance regarding Antiphon the Sophist (c. 480 – 411 BC), to whom T. L. Heath attributes the invention of the **Method of Exhaustion** a method of fundamental import to Mathematics, that initially Eudoxus and later Archimedes developed and perfected, and a method which, according to current-day mathematicians, was the precursor to Calculus and the theory of integration (a matter to be further addressed in the section on Epistemology). However, it is not entirely clear whether this is the person who must be credited with this important discovery, or someone else with the same name, see ref. [19]. Conflicting reports on Antiphon exist from the writings of Thucydides, Aristotle and Simplicius. Another case of a non-listed mathematician is that of Amyntas of Heraclea, purportedly a Plato's student, someone that Proclus mentioned as having perfected Geometry, and having done work on the subject of squaring the circle, along with the brothers Dinostratus and Menaechmus, see ref. [9]. Mathematicians such as him are not listed, due to lack of more and reliable information. Reliable written evidence is a critical issue in the decision to include a name on this paper's List; for more on this important subject the reader is directed to the concluding section of this paper. Parenthetically, on writers such as Proclus, and other Roman and Byzantine Era authors of treatises on Ancient Greek mathematicians and astronomers (such as Aetius, Porphyry, Iamblichus, Diogenes Laertius, Simplicius, etc.) the reader is addressed to ref. [A1]. Finally, another individual not included on the List is Chrysippus of Soli (c. 280 – 206 BC), thought to be an important member of the School of Stoicism established by Zenon of Citium, who was also the founder of the Stoa Poikile (ΠΟΙΚΙΛΗ ΣΤΟΑ) Academy in Athens, see ref. [20]. He is not included in the List because his contribution to Mathematics proper is not considered to be of such significance, although his methodological inquiries and his profound epistemological questions regarding their foundation in Logic were of extreme interest and import, as the field in late 19th century evolved into Symbolic Logic and ultimately to the work by Kurt Gödel, see ref. [A4]. That work will be revisited later in this paper, when the epistemology of Ancient Greek Mathematics will be reviewed.

In addition, more detail review by this author of the work by the 6th century BC Miletus School of Thales and Anaximander, Pythagoras, Eudoxus, Callippus, Euclid, Archimedes and Apollonius (and of course Eratosthenes) is found in ref. [A1]. For each of the individuals on the List, at least two key sources were primarily consulted:

https://plato.stanford.edu/index.html and https://mathshistory.st-andrews.ac.uk/

For the rest of the individuals on the List, not covered in any great detail by these two **primary** sources, two additional and more commonly consulted (but introductory) **secondary** references, that of “Encyclopedia Britannica” and “Wikipedia”, were employed. The second among the two primary sources (that from the Mathematics Department at the University of St. Andrews) lists and mentions these individuals (noting however that they are “Greek”) as born in cities under current countries’ names.

As a consequence, all Ancient Greek mathematicians and astronomers born in the region of Ionia, in Western Asia Minor are listed under the country of Turkey; those of Magna Graecia are listed under the country of Italy; those from Alexandria and Cyrene are listed under the countries of Egypt and Libya correspondingly; numerous individuals born in regions under the Seleucid Empire and later of the Eastern Roman (Byzantine) Empire, but worked during the Hellenistic and Roman periods, are listed under their various contemporary countries’ names such as Egypt, Palestine, Syria, Iraq, etc. Notable is that the bulk of the Ancient Greek mathematicians and astronomers included on the List of 55 were not born on what is currently the political and legal jurisdiction of the Greek mainland; most were born and worked outside the current borders of the contemporary country of Greece.

What all that complex admixture of nationalities and countries, ancient and modern, indicates is that an unavoidable fuzziness persists and characterizes designations and labeling of both ethnic groupings and geographical locations. And that fuzziness is consistent with the cannons of the New Epistemology, of refs. [A5], [A6], [A7].

However, this paper is not aimed at adjudicating and resolving in any sense such convoluted, ethnic-geographic-genetic-cultural-political-linguistic and in general spatial-temporal dynamical (and largely unresolvable) complexities. Hence, the fuzzy, fluid, blurry term “Greek” must be appropriately viewed and interpreted.

A further comment on references that served either as primary or secondary (another fuzzy distinction) sources of information regarding the biographies of the 55 personalities mentioned in this paper (a kind of “Who is Who” list in Ancient Greek Mathematics and Astronomy) is a *sine qua non*. One of the strains of work that the great analysts of the late 19th century (on the basis of which the great works by T. L. Heath, J. L. Heiberg and others, see for more detail ref. [A1], were carried out) is the “Doxographi Graeci” a term coined as part of work undertaken by Hermann Alexander Diels; see ref. [10] about him, and ref. [11] about the key work by Diels on “pre-Socratic” (another term he popularized) philosophers. Since many of the individuals on this List predate Socrates, Diels’ work is of relevance.

Doxography (ΔΟΞΟΓΡΑΦΕΙΑ) is a term meaning "the writing of an opinion" (the term ΔΟΞΑ meaning "opinion", but also "glory", the term "ΔΟΞΑΖΩ" meaning both "espouse the viewpoint" and also "glorify"). It is an ideological, scholastic, as well as scientific and literary source, along with it being the logical and linguistic foundation of analyzing a text's sources. It attributes various qualities, such as authenticity of writing, to specific individuals as well as it pegs chronology among other literary and epistemological attributes to a source. On doxography see also ref. [12].

This "doxography" however is not simply a 19th century legacy and *modus operandi* in the field of Philosophy and Philology. It is a long-held tradition and practice to which writers of a given sociocultural perspective attached (and still attach) to a specific point in space-time, bound in reviewing and interpreting literature originating at and by a prior culture. A new "doxographic" literary tradition is apparently forming in the latter part of the 20th century and the first part of the 21st century in the form of a "Romance" towards the ancients (at least part of them). An example of such doxography is reviewed, analyzed and discussed in the paper by this author cited in ref. [A2], regarding Early Babylonian Mathematics.

The subject of doxography brings up another epistemologically critical issue. Historians were explicit in recording others' opinions (such for example was the work by Diogenes Laertius, see ref. [A1]). They were less forthcoming though, in letting their readers know as to whose opinions they were themselves expressing; as well as what their own ideology, biases and opinions were while recording events and describing various historical figures, see ref. [A3]. It is a matter of interpretation by a reader to infer such biases and imperfections, which are also in turn subject to interpretations and the reader's own biases, viewpoints and perceptions.

On the Demographics of the List

Recording the appearance of the 55 individuals on this List, by the (estimated, and at times grossly) date of birth, one has the following table: in the 7th century BC {**2**}, in the 6th century BC {**5**}, in the 5th century BC {**12**}, in the 4th century BC {**12**}, in the 3rd century BC {Archimedes plus **10**}, in the 2nd century BC {**6**}, in the 1st century BC {**1**}, in the 1st century AD {**3**}, in the 2nd century AD {**0**}, and in the 3rd century AD {**3**}. Of course, date of birth is not the time to pinpoint productive activity as a professional mathematician or astronomer (or of any other closely allied field and activity); however, that complication is left as a possible extension (among numerous others) that could emanate from the 55-name List provided.

It is quite clear then from the above summary table, that: (a) the nadir (that was reached in a sharp increase following the 7th century BC) in the production of Ancient Greek mathematicians and astronomers occurred during the **Classical Era of Greece**, where the two centuries with the highest number of mathematical geniuses (12) were the 5th and the 4th century BC, with the 3rd century BC and the century of Archimedes coming a close second with 11; and (b) past this peak, a sharp decline in this number occurred, with the 2nd century BC dropping to about half, trickling down to almost nil over the next three centuries reaching zero in the 2nd century AD. The ending of this extraordinary streak took place in the 3rd century AD with Diophantus and Pappus.

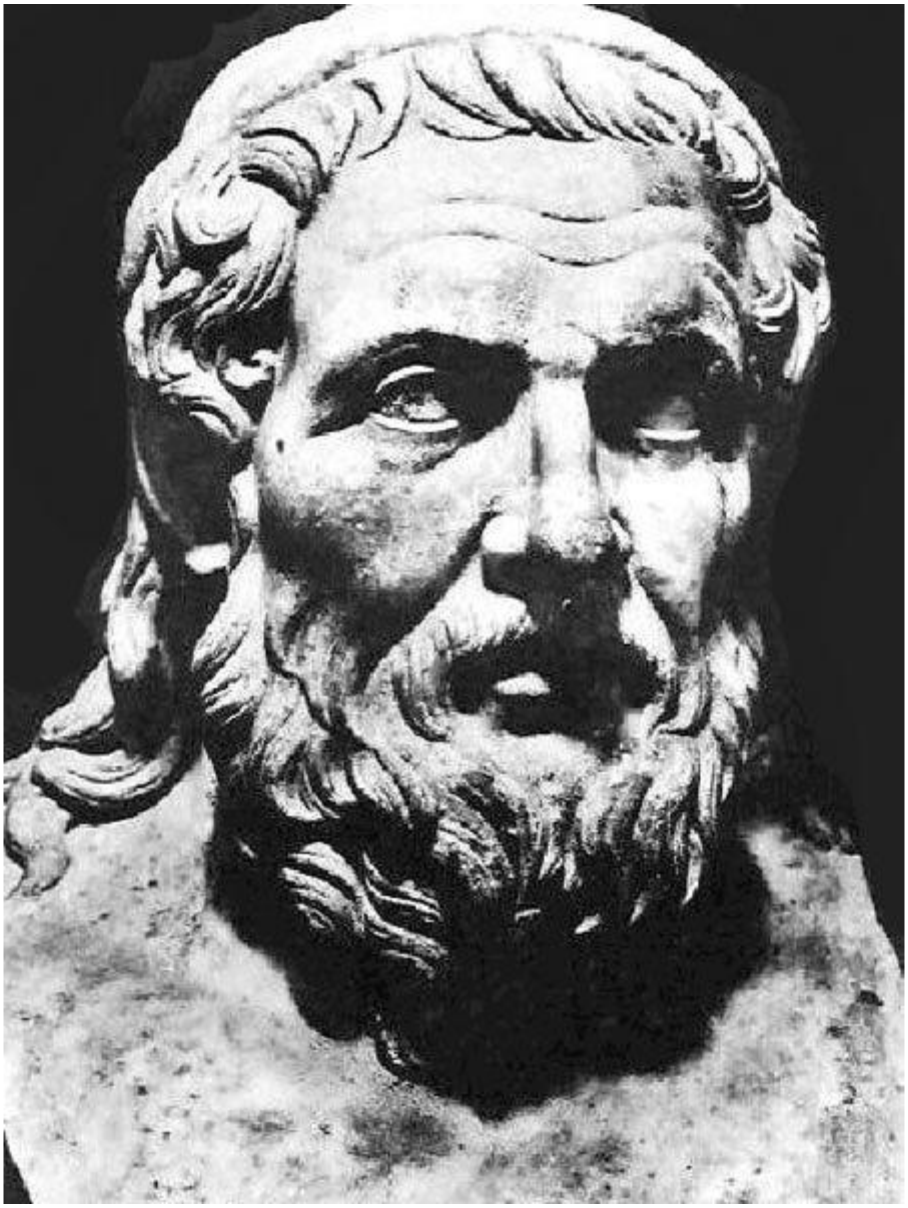

Figure 6. Apollonius of Perga (circa 262 – 190 BC), the "Great Geometer". His seminal, and extant, work in Mathematics on ***Conics***, dealt with second degree curves produced by planes intersecting third degree surfaces (cones). He analytically introduced the notions of (and coined the terms) "ellipse", "parabola" and "hyperbola". This figure stands tall as being the last great Ancient Greek mathematician of the BC period. Source of figure and credit to: ref. [F6].

The brief demographic summary table speaks volumes about the History of the Ancient Greek Civilization, and its demise during the Roman Era and the Roman conquest of the Helladic Space. It demonstrates that the aura of splendor of Classical Greece was highly associated with the presence of the great mathematical minds that in effect provided the fuel and energy for that extraordinary and unique event in Western Civilization, its Arts and Sciences and Humanities.

The role of what is referred to as "the Hellenistic" Era, a transitional period between the Classical Greek period and the Roman as well as Byzantine Periods, is a complex one in need of special (additional) attention, and clearly outside the subject matter of this relatively short summary paper on the great Ancient Greek mathematicians and astronomers. In the next section of the paper, another major intended topic to be covered is presented, the Epistemology behind the Ancient Greek Mathematics.

Ancient Greek Mathematics' Epistemology

This section of the paper will attempt to sketch out the fundamentals of the Epistemology as viewed today, found in Ancient Greek Mathematics, not in Astronomy, and certainly not in Philosophy. Although some ancient philosophical Schools' position on Mathematics (mainly Geometry) will be briefly mentioned, such as the Epicurean and Stoics' views, this section does not intend to cover the much broader epistemological foundations of the various philosophical Schools of the Helladic Space (and there were numerous) at the course of that millennium (from the middle of the 7th century BC to about the 350 AD time frame), which is the focal temporal period of this study. The subject is enormous, as are of course all topics that have been only touched upon by this paper. Hence, only certain among the key selective topics will be presented. Numerous more topics that could be included are not covered for reasons of limited space and time.

The great historian of mathematics, Thomas Little Heath, in his classical writings on Ancient Greek Mathematics refers to these Mathematics as "Science", itself an epistemological issue of great importance. Humbly, this author disagrees with this assessment. Mathematics are not Science, although one must admit that (and in light of the cannons of the New Epistemology) the borders between Mathematics and Science are certainly ambiguous, fuzzy, fluid and blurry. It is however rather clear that in various scientific fields one forms testable hypotheses, with the objective to reject the null hypothesis on the basis of somehow obtained empirical evidence and the subsequent employment of some form of Statistics.

This is definitely what is not characterizing Mathematics, at least not explicitly. Mathematic are based on Logic, and the formulation of axioms, theorems and proofs, obeying internally consistent drawn up cannons of Logic. No empirical evidence is required to support them, as is the case with any Science. Mathematics form Virtual Realities, a subject to be further developed shortly; Mathematics are not Reality. Science, on the other hand, offers hypotheses in efforts to describe and explain Reality.

What Mathematics do, because they are founded on Logic, share with any Science though is Epistemology. This is the area that this section of the paper addresses. And epistemological issues in Mathematics were present and became of essence at the very beginning, with Pythagoras' Mathematics, and even earlier possibly with Thales' Theory of Proportions, when he employed that theory to estimate the height of pyramids in Egypt, see ref. [A1].

The Pythagorean Cosmos

At the outset it must be noted that at both the early as well as late stages of Archaic Greece, it is quite difficult to distinguish between the work by Pythagoras and that by the various Pythagoreans that followed him. It is a fuzzy area of scholarship, where specific individuals' ideas get entangled. One finds them in a superposition of state with ideas by others in a set (also fuzzy) that is referred to now, as well as back then, as the "Pythagoreans". On this issue, the reader is directed to ref. [A1], where Pythagoras' life and work as well as his Society, the group consisting of Pythagoreans and neo-Pythagoreans and associated groupings, are discussed more in detail.

With Pythagoras, one comes across mathematical "proofs", for the first time in the History of Mathematics. To state that this is of extreme import and a ground breaking enterprise, a stroke of genius and a giant leap in Humanity's intellectual development and evolution, would certainly be an understatement. It launched Humanity into a new path, a revolutionary and radically different path, built on explicit and formal **reason** and **truth** rather than row and informal **belief** and **conjecture**. However, both of these types of notions (truth and reason) are now radically re-examined and re-viewed, under the microscope of the New Epistemology, see refs. [A5], [A6], [A7].

Disputes rage at present about the source of the Pythagorean Revolution. No matter however, who is precisely behind these ideas (an unresolvable issue at present from our vantage point) the connection between the Cosmos, Reality and Mathematics (Geometry and Number Theory in specific) pre-occupied all members in that blurry group, Pythagoras and the Pythagoreans. They believed that Nature (Music and Astronomy were fields where they found concrete empirical evidence to justify their claims) could be described (modeled) by mathematical relationships.

Moreover, the Pythagoreans considered that numbers are behind Nature's elements and functions, see refs. [1], [3], [13], [14]. One could argue that Pythagoras' model was the first modeling effort of Humanity to simulate the Cosmos, Nature and Reality. The Pythagoras Theorem, its proof and the associated Cosmology and Numerology, specifically the mathematical beauty of the right-angled triangle formed by the triad (3,4,5); the aesthetic appeal and the embedded harmony of the tetractys (ΤΕΤΡΑΚΤΥΣ, 1+2+3+4=10); and the powerful simplicity of "triangular numbers" and the associate possibly first arithmetic series formula ever established:

$$S[N] = 1 + 2 + 3 + \ldots + N = N(N - 1)/2$$

were the early developmental stages of mathematical modeling of Nature, Society and Reality. One can suggest that the Pythagorean Theorem was the first ever-made mathematically-based epistemological statement about the Cosmos. As discussed earlier, Pythagoras was possibly the first to discover incommensurability, by analyzing the square root of 2. Since then, Epistemology has evolved, of course; it has considerably developed and significantly matured.

The New Epistemology, see refs. [A5], [A6], [A7], has come to espouse the polar opposite of the Pythagorean view; it denies the absolute validity, exclusive presence, and thus far unquestionable capability of Mathematics (geometric shapes) and Numbers to capture the full extent of Nature, Society and Reality as their **exact replicas**.

Mathematics (and by this fuzzy term on implies what is referred to as Geometry, Algebra, Trigonometry, Calculus, and Numbers as well as notions such as zero, infinity, negative numbers and incommensurables) are means that very intelligent and inquisitive analysts and observers of the Cosmos have devised to obtain and record empirical evidence, formulate theoretical propositions, test such propositions, and thus describe by (albeit) approximation, incompletely and always imperfectly the, in such mathematical manner perceived, Cosmos. Nothing more, and nothing less. This is the main epistemological theme of this section, and to be further developed by examining also Euclid and Archimedes and their specific Epistemology.

Euclid and Epistemology

In Euclid's *The Elements* one comes across for the first time, given the currently available record, the logical organization of mathematical notions such as: axioms, theorems (propositions) and proofs. Euclid added a fundamental component to the Pythagorean duo theorem-proof; the Pythagorean duo became the Euclidian trio: axiom-theorem-proof. In The Elements, work by prior distinguished mathematicians, such as the work by Eudoxus, is incorporated, enhanced and streamlined. Consequently, statements about the epistemological aspects of Euclid's work are equally applicable to these prior mathematicians as well.

Euclid, in his thirteen-Book treatise, *The Elements*, introduced, through his own version of a "Preface", **definitions** of terms and **postulates**. It is worth delving a bit on the subject of definitions. Key definitions involve key notions in Geometry, such as: what is a point, a line, and more fundamentally what are the dimensions in Geometry. Points are defined as geometric elements (objects) that can't be divided (the principle of indivisibility) and are dimensionless. Lines are defined over one dimension and have no width. These definitions were in part met with some demur by Ancient Greek philosophers and have forced the mathematicians to devise ways around certain issues that emanate from such definitions, as it will be shown shortly.

Postulates are statements such as that one can draw a straight line between two points, that being the shortest line type one can draw linking the two; and that two parallel lines, i.e., lines forming 90° angles with a line that crosses them, however long they are extended they never cross; and equivalently that parallel planes do not meet, etc. It turned out that these were among the fundamental axiomatic statements of Euclidean Geometry. Relaxation of such definitions and postulates, particularly those regarding parallel lines and planes, brought about many extensions. Examples include the **Theory of Optics**, on a subject in which Euclid and others had written; and the **Theory of Perspectives**, whereby an observer's view from a point off a plane on which two parallel lines have been drawn, as seen in perspective, seemingly violate this postulate; as well as the various version of non-Euclidean Geometries in the 19th century are based.

In digressing a bit for a moment, a few more points will be made on the issue regarding “perspective” and Euclidean Geometry. Once a plane on which two parallel lines have been drawn (in accordance with Euclidean Geometry) is seen by an observer off that plane, i.e., in perspective, the straight lines no longer are seen as being straight if projected onto another plane perpendicular to the observer’s line of sight. This bending, would make the two objects (parallel lines on the initial plane) become asymptote (i.e., converge) to the observer’s straight line of sight. These two lines would acquire a curvature, the more pronounced the curvature the closer the eye of the observer is to the original plane. This proposition, to be referred to as Problem A, will be again revisited in the concluding section of this paper.

Moreover, in Euclid’s *The Elements* one comes also across the notion of a **Lemma**, something of a secondary, sub- or auxiliary theorem. In effect, one discovers through Euclid a nuance between a Theorem (Proposition as Euclid refers to them) and a Lemma (ΛΗΜΜΑ).

As it will be discussed later in this subsection, the very notion of a “theorem”, which today we consider roughly to be equivalent to the term “Proposition”, as used in the Euclidean and Archimedean period, was not thought to be equivalent to, and exactly the same as, a “Proposition” back then, see ref. [15], and [16].

A celebrated example of the blurry difference between theorem and Proposition is that involving Democritus, as a geometer; he “theorized” (ΘΕΩΡΗΣΕ), possibly in the form of a **conjecture**, the widely discussed and analyzed by numerous ancient Greek mathematicians property linking cones and cylinders (the volume of a cone with an equal base and height to a cylinder is one third of the cylinder’s volume); but the person who proved it, in the form of a formal Proposition followed by proof, was Eudoxus, and it appeared in Euclid’s *The Elements*, as Proposition 10 in Book XI, see ref. [17]. The same theorem also appears as Proposition 10 in Archimedes’ Book *On Conoids and Spheroids*, see ref. [5], p. 99, but stated in a slightly greater detail and in a more generalized form.

Furthermore, in Euclid’s *The Elements*, one comes across the notion of “**construction**”, such as the statement referred to as Proposition 11 Book XI: “To draw a straight line perpendicular to a given plane from (any) given elevated point (off the plane)”, see ref. [18], in a form slightly paraphrased by this author with the words in parenthesis. In the work by Archimedes (see ref. [5], p. 3, Archimedes Book I *On the Sphere and Cylinder*, one comes across yet another notion, that of “**assumptions**” (such as “of all lines that have the same extremeties, the straight line is the shortest”). Obviously, this is another shade in the fuzzy and ambiguous distinction between notions such as: axioms, definitions, assumptions, propositions, theorems, lemmas, corollaries, constructions, etc.

The manner these notions are **linguistically** and **logically** defined, and what specific (to an extent idiosyncratic) meaning they carry by a geometer are critical in the Epistemology of Geometry. For an introduction to the conventional Epistemology of Geometry, the reader is directed to ref. [19].

This particular aspect of definitions, and the fuzzy distinctions between linguistically neighboring terms in meaning, will be revisited when Archimedes' epistemology will be reviewed in the next subsection (which also contains "axioms" and "definitions" with an ambiguous distinction). Fundamental in the New Epistemology is the inescapably fuzzy borders among sets, and the unavoidably blurry nature of definitions, hence the weaknesses and incompleteness in the rigor of axioms, and thus limitations to the validity and internal consistency of any system of Logic.

With Euclid we come across the very issues that preoccupied and continue to preoccupy logicians of the 20th and 21st century. In refs. [A5], [A6] and [A7] this author extensively discusses and analyzes these foundational issues within the framework of the New Epistemology. Summarily, Euclid, in his six first Books of *The Elements*, deals with planar (two-dimensional) Geometry, i.e., triangles, rectangles, and circles. In his Book V Euclid streamlines the work by Eudoxus, who did spectacular indeed work that involved motion and thus Dynamics. Euclid presents Eudoxus' theory of proportions and incommensurables, see refs. [1], [3], [15].

Number Theory is incorporated in Books VII, VIII, and IX; while Book X deals with irrationals and the work of Theaetetus; finally, Books XI, XII and XIII deal with three-dimensional geometric objects, polyhedral shapes involving complex conceptions of entangled pentagons, such as the icosahedron, with an embedded symmetry in the way pentagons are incorporated in it.

No matter how fundamental these definitional issues brought up by Euclid may be, however, an even more fundamental epistemological issue lurks within Euclidean, pre-Euclidean and post-Euclidean Epistemology. The issue was brought up by pre-Socratic philosopher Anaxagoras of Clazomenae (c. 500 – 428 BC) and by sophist Protagoras (c. 490 – 420 BC). The concerns they raised apply not only to Euclid's Geometry, but to the entire body of geometers, past and present; and it has to do with the fundamental elements of Geometry, such as the nature and existence of points, lines, shapes, and more broadly dimensions, as well as approximations and scale.

As mentioned, in reviewing Euclidean Geometry and the foundational elements of its structure, one must discuss the issue of its critical components' foundational **definitions**: what is a point, a line, a geometric object. It is right there that a fundamental principle associated with the New Epistemology lies, and requires close attention. For that, the volume by Thomas L. Heath is very illustrative.

In ref. [3], pp: 118-122, Heath quotes from four sources (including Sextus Empiricus, Plutarch, Eudemus and Simplicius) on Geometry's constituent definitions, and it involves no less than the following fourteen individuals and their work on only this particular subject, which one may refer today as associated with **scale**: Democritus, Protagoras, Eudoxus, Aristotle, Chrysippus, Euclid, Leusippus, Xenocrates, Hippias of Ellis, Socrates, Prodicus, Hippocrates of Chios, Antiphon and Archimedes. The central issue is the following: Protagoras is reported as having put forward the proposition to "confute the geometers" regarding **tangents**, specifically that "…no such straight lines or circles as the geometer assumes exist in nature and we shall find that a material ruler does not touch a material circle in one point only", ref. [3], p. 118.

In an imaginary exchange, Heath responds to this challenge, and the issues brought up are of extreme interest, resonating to this day, see ref. [A5]. Heath elucidates us by proposing a response, a reply that is ripe in Epistemology and for epistemological analysis:

"We can imagine Democritus replying to this effect 'it is true that due to the imperfection of our instruments we can't draw a mathematical straight line touching a mathematical circle and hence we can't *visualize* the tangent touching the circle at one point. But, none the less, we can see this with the eye of the *mind*, and we know by force of *demonstration*, that it can not be otherwise'", ref. [3], p. 118, 119.

The epistemological subjects raised by this short, but dense in fundamental concepts, passage and imaginary inter-temporal exchange between Protagoras (characterized as a "sophist", a grade below that of a "philosopher or a sage" in the totem pole of Philosophy) and Heath, via Democritus, are enormous. Matters of theory vs. empirical evidence are involved, **imperfections** of instruments of observation, approximations and scale. But above all, the distinction is brought up between what one **perceives** by senses, versus what one could **mentally conceive**.

And this is precisely the point about Reality versus Virtual Reality and Mathematics. Mathematical modeling, according to the New Epistemology, is a means to **approach** and **approximate** Reality for a purpose and subject to observational imperfections and biases, not to **precisely** (or exactly) **replicate** (objectively copy) Reality, as this is **infeasible**, a chimera (XIMEPA).

Geometric points are dimensionless and indivisible, whereas lines are supposed not to have widths. Physically, i.e., in Reality (Nature, the Cosmos) this is obviously an infeasible set of conditions, that can't be met. Indivisibility does not exist. No matter how thin of a line one can draw, as the scale of that line printed on some medium increases, i.e., the scale becomes finer as one looks at it closer, one encounters the molecular structure of the element used to write. This is neither dimensionless or regular. Hence, such conditions can only be assumed, and they are encountered in models, i.e., mental constructions (as Heath notes). These topics are found at the very core of the 21st century New Epistemology and its internal debates (see ref. [A3]).

In addition, and in a deeper sense, matters of **scale** are involved, see ref. [A17]. More detailed information is revealed as one looks closer (at a finer scale) into the geometric diagram's image, and the lines (curved in the case of the circle's circumference or straight in the case of the tangent) acquire more precision. The additional refinement allows for more accurate determination of the point(s) of tangency, although no resolution can clearly show the unique point and the exact moment of touching. No matter how detailed the image gets, and how fine the scale, the issue brought up by Protagoras would persist. But the Heath point remains also valid, as long as it stays at the mental, not the operational, level. This is where fuzziness interferes, and coexisting Realities (operational and mental) can appear in a quantum state. The issue and its underlying argumentation take the point a step further than the qualitatively similar Reality vs. Artificial Reality condition encountered by the Pythagorean view of the Cosmos.

That was a rough and general proposition, but the 6th century Pythagoras issue now meets with the 5th and 4th century BC refined and more specific issue raised by Protagoras (and others, such as Anaxagoras, but mainly by the Epicureans) about the very nature of Geometry and its ability to exactly replicate Nature.

Subjects involved will come to reach a crescendo by the middle of the 3rd century BC when Archimedes employed novel means of attaining **approximations** to points, curves and surfaces by his ingenious use of Eudoxus' Theory of Exhaustion; his view of very large numbers; his treatment of infinity and infinitesimals, a long-standing issue with the Ancient Greek mathematicians, since Anaximander in the early part of the 6th century BC introduced the term ΑΠΕΙΡΟΝ into the Greek vocabulary of analysis. For a more extensive discussion and citations on these very mathematically involved issues, see ref. [A1].

Archimedean Epistemology: infinitesimals, approximations, ΝΕΥΣΙΣ

A confession and acknowledgement are necessary at this point: It certainly is a high honor and great privilege, as well as humility, for anyone to be writing about Archimedes, possibly the brightest mind of Antiquity. Any existing superlatives and adjectives indicative of greatness in Greek, English or any other language, and in combination, are certainly inadequate to capture the full grandeur, magnificence and aura of this unique mind. With all due reverence, this author humbly dares address some of the innumerable and profound contributions Archimedes has made and insights he has added to the total human stock of knowledge. He demonstrated that a real genius makes contributions to a host of fields, inventing some too along the way. The fascinating set of extant Archimedes Books, and the penetrating insights embedded in it are not just and simply Mathematics, Mechanics, Physics, Logic, Philosophy, Epistemology or any other branch of human endeavor. It transcends any and all of them. His work hits at the very core of Arts, Science and Mathematics, as formed then and have evolved (and are still evolving now).

Having stated the above short and introductory comment and paid the necessary tribute to possibly the greatest mind of Antiquity, trying to capture only some of the multiple contributions Archimedes has made (and still is making) to Mathematics and Science, it is the intent to warn anyone to not underestimate the vigor, depth and sharpness of his insights. It is within this framework that the following very brief remarks are made about his work. In Archimedes' work, we come across another term associated with the notion of a proof, nuancing this notion even further. A number of times Archimedes employs the term (translated according to Thomas Little Heath as) "it is **manifest**" (evident); see for example in ref. [5], the cases mentioned on p. lxi; and particularly on p. lix. There, three propositions (they are listed under the heading "Proposition 11" in the book *On Conoids and Spheroids*, see ref. [5], p. 122), on the sections of a spheroid by a plane parallel to the axis are said by Archimedes as having proofs that are "manifest" (evident). It is worth expanding on this expression "manifest" as certain key points emerge by reading the relevant texts. It turns out that the term is subject to interpretation as to its exact, i.e., as intended by Archimedes (if such conjecture can be accepted in face value at present) meaning.

In the book by Johan Ludvig Heiberg (see ref. [21]), where the Greek (along with the Latin) text appears, the expression “it is manifest” (evident) is mentioned in Greek for the same section of the Archimedes Book ΠΕΡΙ ΚΟΝΩΕΙΔΕΩΝ ΚΑΙ ΣΦΑΙΡΟΕΙΔΕΩΝ p. 342 (as in T. L. Heath’s ref. [5], p. 122) “ΤΟΥΤΩΝ ΔΕ ΠΑΝΤΩΝ ΦΑΝΕΡΑΙ ΕΝΤΙ ΑΙ ΑΠΟΔΕΙΞΕΙΣ” (written by this author in uppercase Classical Greek, and not in lowercase Byzantine Greek and as “ΑΠΟΔΕΙΞΕΙΣ” not “ΑΠΟΔΕΙΞΙΕΣ” as it appears in Heiberg). The literal interpretation of the expression “ΦΑΝΕΡΑΙ ΕΝΤΙ” is “they are obvious”.

It is with the explicit purpose of noting this minor difference between the Heath and Heiberg texts, as an indication of some also major differences that exist between the two texts. For instance, in Archimedes’ same Book, Proposition 8, the Heiberg translation of Archimedes’ text and diagram is very different than the corresponding Heath translation into Latin from Greek and in the corresponding diagram. Heiberg has Archimedes proving the proposition following in part Euclid’s *The Elements*, Book V, Proposition 22, as well as Apollonius (Book I, Proposition 1). Heath bypasses both in his exposition of Archimedes’ proof.

Hence, and by going back to the expression “it is manifest” (evident or obvious), a number of things become a bit clearer. Of course, that “manifestation” rests with the eyes and mind of the beholder, as T. L. Heath in an elegant manner insinuates (along with admitting that they do); and the differences between the Heath version and the Heiberg version of Archimedes’ texts points to the manner ancient texts are modified by either translation or copying. Parenthetically, in certain instances one detects clear errors in the Latin translation from Greek by Heiberg as well in the drawing of diagrams; for example, see Proposition 3 in the Book *Measurement of a Circle*, (in Latin “DIMENSIO CIRCULI”, and in Greek “ΚΥΚΛΟΥ ΜΕΤΡΗΣΙΣ”).

Bypassing for a moment differences in letter designations of points within geometric diagrams between Heath and Heiberg, one comes across instances where: the diameter in the figure by Heiberg is erroneously drawn (correctly drawn in Heath); Heiberg inserts equalities where inequalities must be designated, such as conditions (1), (2) in Heath corresponding to conditions at the top of p. 265 in Heiberg, involving the square root of 3; and a condition that should hold as inequality (involving the ratio of a right-angled triangle’s square of its hypotenuse over the square of its base, 349450/23409) appears as equality in Heiberg, whereas it is correctly identified as inequality by Heath; as well as omissions and additions by either Heiberg or Heath in these two texts, see for comparison and as examples Propositions 5, 6, 7 of the same Book.

No matter these differences, however, between Heiberg and Heath (and any possible errors) the more basic issue involved here is of epistemological concern: it has to do with **completeness** and **uniqueness** in proofs, i.e., whether such completeness exists; and, in a larger scope where multiple proofs may be devised of a theorem, whether there exist **perfect** proofs, and the criteria on the basis of which such perfection can be ascertained, assessed and achieved. This is a topic that this author has addressed more extensively in ref. [A6], and particularly in the case of Kurt Gödel’s proofs of his two Incompleteness Theorems.

One of them, has to do with Gödel's "Second Incompleteness Theorem", as offered by David Hilbert and Paul Bernays, which was presented in about 70 pages, see ref. [A6], p. 97. The (open) question is whether such a proof can be complete; or more exactly, what is precisely meant by "completeness". That notion basically rests with the stated axioms and their foundation on some formal Logic and Linguistics, see refs. [A6], [A16].

The subject of multiple proofs to a single theorem (or Proposition) is very clearly shown in the Archimedes' work, when one reviews Proposition 4 (cutting a sphere by a plane so that the segments' volumes are proportional to a given ratio, in effect it involves finding the positive roots of cubic equations by means of intersection of conics, i.e., parabolas and hyperbolas) in the Book II *On the Sphere and Cylinder*. There, no less than three other mathematicians (with three different proofs) are involved, besides Archimedes (and of course, that implies the T. L. Heath's version of Archimedes' proof, as it is from copies of the original that Archimedes' work has survived – the original in not extant), namely: Eutocius, Dionysodorus and Diocles, see ref. [3], pp:62-79. On Dionysodorus' interesting work, in estimating the volume of a torus, as well as in forming an innovative shape of a sundial, see refs. [24], [25].

Before venturing into two other major contributions made by Archimedes in Mathematics (as well as in Statics, Mechanics, Hydrodynamics and Mathematical Physics), on infinitesimals (closely linked to the world of incommensurables) and the Method of Exhaustion", the notion of "Neusis" (ΝΕΥΣΙΣ) he introduced becomes impossible not to mention. In his wonderful book on Archimedes, ref. [3], Chapter V, pp: c – cxxii, T. L. Heath dedicates an entire section on the notion. It is a complex notion, literally meaning as a verb (in singular present it takes the form "ΝΕΥΩ") *to verge* (or a more illustrative but synthetic form, to converge); but all Greek verbs (and words in general) had and have complex etymology and meaning, due to the manner they evolved.

Even though Heath dedicates the chapter on this term, and meticulously explores the cases where Archimedes used it, it still remains fuzzy as to how Heath settles on a singular English word in interpreting the manner Archimedes employed it. Heath always wondered how did Archimedes (and in fact all Ancient Greek mathematicians) ended up in presenting the Propositions and their proofs in such a crisp, clear, exact and efficient manner; he lamented that these bright minds did not provide any hints as to how they derived their impressive, magnificent compositions in both at the moment of creation or during the process of proving them. They did not supply any hints as to the possible labyrinthic paths they took, before settling on what they presented in their treatises. However, some of these questions were to an extent answered and the perplexity partially relaxed, when Archimedes' treatise on *The Method* (a book that will be shortly discussed) surfaced at the beginning of the 20th century.

It is at the core of the creative process, the time of ΕΜΠ-ΝΕΥΣΙΣ, the moment of inspiration and creativity, that one must look for determining the **approximate** meaning of the term. There is **no** exact meaning, as the precise meaning it had, at the time Archimedes used it, is long gone and irretrievable. Archimedes could use any synonym (such as the verb to verge, ΚΛΙΝΩ) to ΝΕΥΩ. But he chose ΝΕΥΣΙΣ to ΚΛΙΣΙΣ or ΡΕΥΣΙΣ, when he employed the term.

Being extremely careful in their use of words, the Ancient Greeks used terms with a prevailing at the time meaning that took them the closest possible to their intended and highly specialized but always approximate meaning. Parenthetically, the term was also used by Apollonius of Perga and Pappus, as Heath notes. To this author, the closest one might get to the intended meaning of the term ΝΕΥΣΙΣ as used by Archimedes at the time he employed it is that of ΕΜΠΝΕΥΣΙΣ, a subject itself of epistemological interest. That may become clear by looking closely at the first instance examined by Heath (on p. ci, in ref. [3]), when one comes across the term in Archimedes' Proposition 5, with his Book *On Spirals* for the first time.

There are numerous such fascinating, straightforward instances, and at the same time aesthetically beautiful in their simplicity examples; some of them are discussed by Heath. In the demonstrative example chosen below, one is asked to take as more or less given, and without any proof, the following statement (slightly paraphrased by this author):

"From any point A on the circumference of a circle, and given any circle's diameter BC, a straight line **can be drawn** from A, intersecting the circumference at point P, and the (extended) diameter at point R, such that the length AP is equal to that of PR".

In effect, this is a statement that can be interpreted as a combination of a "construction", a "theorem", an "hypothesis", and a "conjecture", all admixed. It is an ΕΜΠΝΕΥΣΙΣ, a truth, a point that is self-evident, an irrefutable mathematical statement that apparently is not in need of a formal demonstration. It is the moment when a bright idea struck the mind of a mathematical genius, at an instance of creativity. Such (plentiful in fact) statements on exactness and precision, however, one comes across only in Ancient Greek Mathematics, and possibly in all Mathematics that followed since then, but not in the physical world of Nature, Reality, or the Cosmos.

And while Heath was wondering on how did these great minds come up with these wonderfully and efficiently stated Propositions and proofs, that is as to what were their intermediate steps, and in one of the most important finds of the 20th century, Archimedes' Book on *The Method of treating mechanical problems* (in the form of a letter addressed to Eratosthenes) shows up (*via* some detective work by Heiberg), when everyone was under the impression that this book was no longer extant.

The turbulent life of an apparently original copy of Archimedes' set of seven manuscripts (as always written in the Doric dialect), including *The Method* (in Greek ΕΦΟΔΟΣ or ΕΦΟΔΙΟΝ or ΕΦΟΔΙΚΟΝ) a copy made by the architect of Constantinople's Basilica of Ayia Sophia Isidorus of Miletus circa 530 AD (i.e., approximately seven centuries after Archimedes wrote the manuscripts involved) is chronicled (in a very rough outline, and in need of far more detail) in ref. [22]. Most of the material in *The Method* (translated into English from the Byzantine Greek of Isidorus) is found in T. L. Heath ref. [23], with an Introduction by Heath. There have been more recent updates on *The Method*, with a few more pages having been read, see refs. [32] – [35], where various aspects of the process by which these additional few pages at the end of the manuscript (from the 530 AD copy and in palimpsest form) have been retrieved are presented.

Among various topics one encounters there are subjects such as how Archimedes arrived at the quadrature of the parabola and the cubature of the circle, in ways that carry immeasurable epistemological interest. Archimedes distinguishes between: (a) means that may be **sufficient to suggest the truth of theorems**, although **not** furnishing the precise analytical proofs (ΘΕΩΡΕΙΝ); and (b) the rigorous demonstration of them by **imputable geometric methods** that must be followed before being considered as having been "established" (ΑΠΟΔΕΙΚΝΥΝΑΙ); ref. [23], p. 7.

This nuance in the admixture alluded to earlier is one of Archimedes major contributions to Epistemology, both of that period and of today. Moreover, the Mechanics involved in supplying the intuitive (ΝΕΥΣΙΣ) approach which led him to the statement of numerous Propositions in the field of Statics, Hydrostatics, Hydrodynamics (involving the notions of **static and dynamic equilibrium**) and in effect the first ever treatises on **Mathematical Physics**, he (Archimedes) did not consider as falling under category (b) above. It is the first time that the issue is brought up, as to the **separation** of Mathematics from Science (in this case, Physics).

But by carrying out this Statics related ΝΕΥΣΙΣ, Archimedes gained a foothold into infinitesimals and the infinite is a way that is both different from and equivalent (and in any case not identical) to today's Calculus. It was nothing short of a brilliant move that demonstrated its powerful impact on both Geometry (Mathematics) and Statics (Physics). One can see this brilliance, by taking a close look into Proposition 1 of *The Method*, as insightfully presented by T. L. Heath in ref. [23], pp: 15-18. Through this method, Archimedes produces two equivalent shapes, one of which the geometer knows how to **precisely** compute the area (a triangle); and then he proceeds to making it equivalent, by appropriate scaling and through application of **static equilibrium** (and the Theory of Proportions), to the area of the shape to which one wishes to estimate (in this case a parabola's sector).

Following Heath's notation, the Proposition goes as follows: Designate by ABC a segment of a parabola bounded by straight line AC and the arc ABC. Point B on the arc is where, if a line is drawn from the middle point D of line AB, parallel to the parabola's axis, intersects the arc ABC.

The Proposition is that: parabola's segment ABC has an area which is in a 4/3 ratio to the triangle ABC's area. How Archimedes goes about demonstrating the validity of this argument is nothing less than remarkable.

By his method, which is based on static equilibria (i.e., Statics), Archimedes finds for each and every (thus all) line(s) inside the area of the known shape (the triangle), the equivalent (and thus all) line(s) of the shape the area of which is sought after. And **here we are at the very core of the Archimedean (alternative to integration) way to compute areas**. The method does not involve infinitesimals directly (as Calculus does); but it involves a **complete** and **exact** accounting of all lines (not the *dx* of Calculus), smartly avoiding to list all of them.

In fact, locating an equivalent pair of straight lines in both shapes was enough. Specifically, finding a link between straight lines OP (inside the parabola's segment bounded by straight line AB) and OM (inside an overlaying triangle AFC) are enough elements to prove the Proposition's validity.

The two (parabola's segment ABC and inscribed triangle ABC) are balanced off in an Archimedean construction along a line (lever) HC, with center of gravity at point K, see the figure on p. 16 of ref. [23]. Appropriately scaled, the overlaying AFC and the inscribed ABC triangle, are in a 4/1 ratio; whereas the overlaying triangle AFC is to the parabola's segment ABC in a 3/1 ratio. Then, one obtains directly the sought-after result, namely that the ratio of the two areas, i.e., the parabola's segment ABC is to the inscribed triangle ABC in a 4/3 ratio. On p. 17 in ref. [23], the essence of the Archimedean method is stated (as translated by T. L. Heath):

"***And, since the triangle CFA is made up of all the parallel lines like MO, and the segment CBA is made up of all the straight lines like PO within the curve, it follows that the triangle, placed where it is in the figure, is in equilibrium about K with the segment CBA placed with its centre of gravity at H***."

The essence of the entire exposition (that can be characterized as the admixture alluded to earlier in this subsection on Archimedes) comes at the end of this Proposition, when Archimedes states (as translated by Heath) the following:

"*Now the fact here stated is not actually demonstrated by the argument used; but that argument has given a sort of indication that the conclusion is true*".

It turns out that he had "published" the full geometric proof to this Proposition, in his *On Conoids and Spheroids* Book.

The bottom line is this: Archimedes' method is not the same as that employed by contemporary Calculus. It is a **different method**, that it could be argued it is more efficient than that of Calculus. Moreover, Archimedes did not **need** Calculus to prove his Propositions. Given his intent to always be **exact**, he avoided the types of **approximations** one comes across in Calculus, with the employment of limits and *dx*. Thus, the claim made by certain contemporary mathematicians that he was either an "inventor" of Calculus, or a pre-cursor to Calculus are not exactly accurate. One may argue whether Archimedes' method is inferior or superior to Calculus; it depends on perceptions, and personal biases, according to the New Epistemology's canons.

But it is not Calculus. One might say that if Archimedes would be alive today, he would not have adopted Calculus as a method to deal with the types of problems he dealt with back then.

Archimedes other signature method was **The Method of Exhaustion**. It was not invented by him, that distinction belongs apparently (notwithstanding Antiphon) to the other great Ancient Greek mathematician Eudoxus of Cnidus. But Archimedes elevated it to an Art, in employing it to derive not only approximations to irrational numbers (such as π) but also to find the square roots of very large numbers. He also used it to derive the length of circles' circumference and the surfaces of spheres. Here is another facet of how the infinitesimals and approximations were used by Archimedes, in contradistinction to how they are used in contemporary Leibnitz-Newton type Calculus. Archimedes found the method to **inscribe** and **circumscribe** equal-sized **polygons** to geometric figures (circles or spheres), set any arbitrary limits (in the form of setting **upper** and **lower bounds** in lengths of circumference in the case of circles, or surfaces in the case of spheres).

Archimedes used this method to find a **better approximation** to either the length of the circumference or the area of the circle, or surface of the sphere, by approaching it on both sides (upper and lower bound) to any **arbitrarily set initial conditions**. This is the method used throughout Archimedes' Book I *On the Sphere and Cylinder*.

What is very noticeable in the above Archimedean process is that he does not resort to deriving an **average** between the upper and lower bounds obtained. That average would have automatically produced a better approximation to either of the two bounds. But that average, which would be a polygon that would be intersecting either the circle's circumference or the sphere's surface, would be a peculiar object, neither fully inscribed or circumscribing these two geometric objects. It would be an ambivalent object, a polygon bearing a fuzzy connection to the two geometric shapes. And ambivalence was not one of the attributes that the Ancient Greek mathematicians wanted to entertain. This issue, to be referred to as Problem B will be revisited in the concluding section of this paper.

It is reasonable to assume that Archimedes was well aware of that, it is simply too simple not to think about it. Yet, he does not even mention this option in any of his Propositions. That omission leads one to assume that he considered this type of "statistic" not to be pure Geometry, and a perversion of his ingenious method. And this likely hesitancy on his part probably was due to the fact that the average would **not** have been known if it is less or greater than the sought-after quantity (length or the circle's circumference or surface of the sphere). Hence, it would not have been good enough for the Ancient Greek mathematician.

Statistics is a branch of Mathematics that the Ancient Greek mathematicians never considered. Of course, they were familiar with "mean proportions" early on, certainly by the time of the Pythagoreans, with the Law of Proportions having been established in Ancient Greek Mathematics at the time of Thales, who estimated the height of pyramids on that basis, see ref. [A1]. The severe limitations, imperfections, and incompleteness of this branch of Mathematics, as it is used in contemporary Science, has been discussed within the context of the New Epistemology, and it is addressed in more detail in ref. [A5] by this author.

There are numerous other epistemological issues that Archimedes' work brings about, and echo to the present; they are to some extent addressed in ref. [A1] by this author. Besides the issue of exactness and approximations a closely related topic is that of discreteness and continuity (especially in the context of Nonlinear Dynamics, and dynamics in general). This epistemological subject has been addressed in many of this author's papers in the past, that the reader can access in the cited author's references here. This issue, in combination with divisibility and infinitesimals, goes way back and predates Archimedes. It was addressed also by Democritus, see ref. [16], Anaxagoras, see ref. [28]. It had preoccupied the Ancient Greek mathematicians and philosophers, in its connection with the notion of infinity (ΑΠΕΙΡΟΝ), especially since the latter's apparent introduction into Greek Philosophy by Anaximander in the 6th century BC. The debate was present in Aristotle's *Metaphysics*, see ref. [29], section 12, among other philosophical Schools in the 5th to the 3rd centuries BC.

Parenthetically, and among the numerous contributions Aristotle made to Epistemology (however, not the subject of this paper) he also discussed the nuances among fundamental definitions of foundational mathematical terms, such as “definitions”, “propositions”, “postulates” and “common notions” (a topic already discussed in earlier parts of this subsection in reference to both Euclid and Archimedes), see ref. [30].

The fundamental dichotomy in Dynamics of Nature, Society and Reality (the domains of the New Epistemology) between discrete and continuous change was, is and will remain an issue of critical epistemological concern. Infinitesimals and infinities, very large and very small in **scale**, and how the mathematician can (or should) approach them, are encountered in the work by Eudoxus, see ref. [31], work that is more extensively covered in ref. [A1] by this author; and possibly by Antiphon, see ref. [19]. The issue of infinitesimals comes fully to the forum and culminates, as already discussed, with the deeply influential and foundational Propositions in all Books of Archimedes in which, one way or another, the topic is addressed.

Another major clause of the New Epistemology is that in Nature, Society and Reality the condition involving **equalities** does not exist, it only appears in the world of Mathematics and in Virtual Reality models. Hence the conditions involving **inequalities** are of central import as well. Archimedes did not apparently confront this issue head-on, although he may have been well aware of it, it is simply too risky to presume that Archimedes was unaware of fundamentals in Epistemology. Thus, although the Ancient Greek mathematicians, including Archimedes, did not exactly deal with this epistemological issue, they introduced ways to deal with inequalities; Archimedes’ methods of assuming inequalities (in effect, setting up a Theory of Inequalities) and then rejecting them as leading to *reductio ad absurdum* conditions was pioneering in this regard.

Archimedes’ work is not dead, but still very much alive, open to growth and extensions. Subjects of epistemological interest that he, along with the other great Ancient Greek mathematicians and astronomers mentioned in this paper, brilliantly raised still resonate to this day.

Conclusions and some comments on evidence, geometric perspective and ambiguity

Numerous subjects were briefly touched by this paper; central however were two: the drafting up of a List of 55 outstanding Ancient Greek mathematicians (in part also astronomers and in part, unavoidably, philosophers), almost all amply documented by History’s written record, as an indication of an extraordinary density of intellectuals who lived at a time period stretching from the Greek Archaic times to pre-Byzantine Era in the Helladic Space, and made significant contributions to what is considered to be the contemporary field of Mathematics. The second major topic of this paper (which is part of a larger study) was to look into certain foundational epistemological aspects of Ancient Greek Mathematics, especially as reflected in the work by Pythagoras and the Pythagoreans, Euclid and Archimedes. In so doing, basic epistemological topics were addressed found in the work of these brilliant minds, topics that resonate to this day. Their work is not dead, but still pretty much alive and capable of evolution.

One of the foundational issues addressed by the New Epistemology is that associated with the existence, presence at some time, perception, interpretation and reporting of **evidence**. Hence, the matter of evidence on evidence, and the possible infinite regression associated with this subject, is a central matter of concern for Epistemology. There is enough written and relatively well documented **extant** evidence to suggest that much of the written work by Ancient Greek mathematicians and astronomers that existed at some point in the past, in some form or another (original or copied), has been lost. It is beyond simple “alleged” evidence.

Their relatively undisputed prior existence is established by references to such no longer extant work, as for example the work by Eudoxus, whose entire repertoire of books albeit lost to History has survived in the historical record, by other writers, who referred to their work. Archimedes for example, refers to Eudoxus’ work in the epistle attached to his Book I *On the Sphere and Cylinder*, by which Archimedes submitted his volume for publication to some Dositheus (likely at the Great Library of Alexandria), see ref. [5], p. 2.

This type of evidence is quite different than the alleged “evidence” put forward by claims involving mostly idiosyncratic hypotheses by current day analysts about Neolithic monuments, for example by the so-called “archaeo-astronomers”, who currently supply commentary on Neolithic, Bronze and Iron Ages monuments, contending that the ancient architects, engineers and builders of these monuments “knew” about advanced subjects in either Mathematics or Astronomy or even Trigonometry. Such indirect, highly speculative and controversial “evidence” is not admissible in this case, and it is not consistent with the standards on evidence employed in the drafting of the List supplied in the main text. There is a huge difference between no longer extant but formerly existing evidence on the one hand, and total lack or absence of evidence on the other. The motto, also used by the SETI (Search for Extra Terrestrial Intelligence) outfit claiming that, “absence of evidence is no evidence of absence” is a sophistry.

In this summary section of the paper, two Problems identified in the main text will receive a bit of more detailed albeit still brief attention. Problem A is related to the issue of **perspective** in Geometry and also in Dynamical Geometry. Problem B is associated with the subject matter of using unavoidable incompleteness, imperfections and **ambiguity** one encounters on both fronts, evidence and theory, in addressing subjects related to exactness in formulating theory, examining empirical evidence, obtaining measurements or setting up and following a system of Logic in Science and Mathematics.

Problem A. **On Perspective**. Perspective as a subject is of course immense, see refs. [36], [37], [38] for just a glance on it. It is related to Relativity, another immense and interdisciplinary topic in both the Natural and Social Sciences. In Geometry, the topic goes back to the Ancient Greek mathematicians, and the issue is reviewed in ref. [36]. In Architecture, when 3-d structures are drawn on 2-d drawing surfaces either Euclidian Geometry (where parallel lines do not cross) is used to produce **axonometric** diagrams; or Perspective Geometry is employed, involving a plane of reference and a point of sight obeying non-Euclidean Geometry (whereby parallel lines converge towards a particular point of sight onto the plane of reference).

In Painting, and following the Renaissance, non-Euclidian perspective was extensively used to structure compositions on canvas, following a non-perspective embedding Byzantine Era Painting. The author has brought up the issue of perspective in the context of viewing sculptures, in ref. [A15]. This brief elaboration on the topic is an extension of that work, and it involves a novel concept (as alluded to in the main text).

A **simple assumption**, or a way that is **not close to Reality**, an analyst can make to approach the topic of perspective is to assume that two parallel lines drawn on a plane (to be designated as P1) obeying Euclidian Geometry, when seen from a single (by a one-eyed observer) point **off** the plane (no matter the location of the observer off the plane but within a three-dimensional Cartesian space), **would converge but not cross** no matter their length at some **end point**. Stated slightly differently, the two lines **will be perceived (or "seen") by the observer as if** they converge to a very distant end point (a point that is called "infinity" in conventional modern Mathematics). In a dynamic perspective, both eye (and hence end point) as well as reference plane (the horizon, or the height at which the observer's eye is located above ground level, or P1) change as the observer moves in the Cartesian 3d space. In Static Geometry the observer's eye remains at a constant location immobile within the Cartesian space, as is the original plane P1.

Moreover, if a new plane P2 is drawn, also off which the one-eyed observer is located, **perpendicular** to the observer's line of sight and at some distance **D** from the eye, and onto which the P1 drawing of the two parallel lines are projected, then one encounters a **geometrical ambiguity**. If, under this **simple assumption**, one is to trace on P2 the straight lines of P1, as seen from the single eye of the observer off P1 and at a distance **D**, then these seemingly (i.e., assumed to remain) straight lines, although parallel on P1, are no longer parallel on P2. The ambiguity is resolved by the assumption discussed next.

A **complex assumption**, or a way much closer to Reality, to approach the subject of perspective is to consider that under the perspective of an observer off P1, the straight lines of P1 do not remain straight lines when projected onto P2. Instead, **they curve**; and as mentioned in the main text, the curves asymptotically converge to the observer's line of sight as projected onto the P2 plane. If the line of sight is on a plane P3 vertical to P1, and the eye of the observer is at a point on such a plane that is positioned to run parallel to and someplace **between** the two parallel lines, and the end point of the observer's line of sight points at the infinity point, then the two straight parallel lines as drawn on P1 undergo **a switch** on P2 becoming both concave (i.e., facing away from the observer's eye), tending (asymptotically converging) towards the infinity point onto both P1 and P2. However, if the line of the observer's sight is off such a plane P3, then one is seen as convex, whereas the other as concave (facing towards the observer's eye).

In ref. [A15], this author put forward the proposition that under Euclidian perspective (simple assumption) any ellipse on plane P1 can be viewed as a circle on another plane P2 (both established as previously stated) and *vice versa*. This would not hold under the complex assumption, as the closer to the observer's eye section of the ellipse would be seen as a having a more pronounced curvature than the section of the ellipse further away from the observer.

Put differently, the cone formed by the observer's eye and the circle (on either P1 or P2) would not be the Euclidean cone (on either P2 or P1). The issue requires further analysis, as novel shapes are involved when a circle or ellipse (or any 2-d shape) is viewed from an angle, at a distance, and under non-Euclidean Geometry in perspective. And similarly, is the case when solids are viewed under a non-Euclidean perspective.

Problem B. **On ambiguity**. At this stage of the concluding remarks a few points will be raised regarding the issue of fuzziness and ambivalence regarding geometric shapes. The issue was brought up in the main text when the possibility by Archimedes was there to deal with the upper and lower bounds involving the potential to average the circumscribing and inscribing polygons' lengths to obtain an estimate of a circle's circumference or a cylinder's surface (or the surface of any other solid, be that a cone or a sphere or a torus) in Archimedean Geometry. It will be discussed within the context of the New Epistemology, where fuzziness plays a foundational role.

The issue is touched again simply because it could be fruitfully employed in analyzing Archimedes' method of exhaustion in proving theorems. While in the main text, the significant advantages the method holds (in comparison to, for example, Integral Calculus) for getting closer to either a 2-d (a circle's circumference measurement) or 3-d (sphere, cylinder, cone, torus) geometric object's surface at any desired level of approximation (i.e., some arbitrarily chosen upper and lower bounds), but where the issues of ambiguity and fuzziness in the object's curve or surface itself were not addressed.

A circle or a sphere are thought in conventional, Euclidian or non-Euclidean, Geometry to be continuous, i.e., no gaps or steps or discontinuities are assumed to be present in their form. This might be a convenient assumption in Mathematics, i.e., Virtual Reality, but in Reality, such abstract and smooth curves and surfaces do not exist. On the topic of continuity and the associated infinitesimals, see ref. [39]. As argued in the cited papers by this author on the subject of the New Epistemology, Nature and Reality are foundationally discontinuous ecologies with objects and events marred by discontinuities (as some scale).

Archimedean methods of exhaustion, based on polygonal type approximations to curves and surfaces may be a means if not to actually counter this ambiguity and fuzziness in the fundamental composition of Nature and Reality, but at least to improve on approaching it more effectively. The "average" between the inscribed and circumscribed polygon is not a statistical tool, but a more complex "hybrid structure". The interesting part in this angle of the issue is that such an insightful and brilliant method, as applied by Archimedes (and Eudoxus, and potentially Antiphon prior to him), if extended along these lines, may be able to bypass the discontinuities related difficulty by approximation. **Archimedes' Mathematics well has not run dry yet**.

And it does so, by using Mathematics the cornerstone of Virtual Reality modeling. However, these Mathematics open up new avenues for work, new hybrid geometric figures and new methods not viewed and considered thus far by mathematicians, either ancient or modern. These suggestions for further research bring to completion this paper.

Figure 7. Pappus of Alexandria (c. 290 – 350 AD). He was the last great Ancient Greek mathematician of Antiquity. Pappus is known for a number of significant contributions to Geometry. His ΣΥΝΑΓΩΓΗ (or, in English, "Collection"), which includes the Pappus' **hexagon theorem in Projective Geometry**, are extensively covered by ref. [3], pp:434-463, where all of Pappus' eight Books are analyzed. In ref. [26], a very brief summary of his life and work are presented; and in ref. [27], his "hexagon theorem" is briefly discussed. Source of figure and credit to: [F7].

Note 1. On the then relative population size of the Helladic Space

It was an extraordinary and unique period in humanity's existence, that millennium-long time frame's endeavor, as it unfolded in the Helladic Space. A burst of creativity and an explosion in human knowledge and understanding of Nature and Reality occurred then and there. It resulted in the germination of a mature Culture, and the formation of a robust Civilization built around the individual. That historic experiment, which culminated within Athenian Democracy, experienced unprecedented Dynamics. Drastic as well as dramatic Evolution in all facets of the Arts, Sciences, Humanities and the Professions ensued. Hence, it is logical to search for, survey and speculate as to the underlying conditions that led to that sudden and fundamental transition, a switch at the threshold of the Greek Archaic Era following the Greek Dark Age.

The purpose of this **Note 1** is hence twofold: first and mainly to highlight the relatively high **spatial-temporal density** of the 55 mathematicians and astronomers of the List offered in the paper's main text; and second, to very briefly address the topic of an apparent sudden and significant increase in the population of the Helladic Space during the transition from the Greek Dark Age to Archaic Greece. Obviously, population changes alone are not the only factor that contributed to this spectacular transition. Other factors must have played a role, significant or minor is next to impossible to trace, three millennia past the cosmogonic event.

The New Epistemology, and especially its Fuzzy Sets Theory; as well as the Nonlinear Dynamics; the methods of superposition of states as drawn from the field of Quantum Mechanics; and issues associated with evidence and measurements lurking in the field of Modern Metrology; these four areas being the foundational cornerstones of the New Epistemology, prevent anyone from offering definitive answers to this question and supplying a full description of the critical transition that saw the Ancient Greek Mathematics (and Astronomy) passing the threshold from the Greek Dark Age to Archaic Greece, from pre-History to History.

The exact spatial-temporal detailed (disaggregate) Demographics of that temporal period, spatial expense and critical transition are of course not known, and neither could they be. It is only in the form of gross estimates and with an unknown degree of validity, accuracy and confidence, hence highly **speculative**, that the total population of the geographical settings to which this study refers in the c. 750 BC – 300 AD time period, one can restrictively and cautiously work and tentatively gauge.

There are some references which, based on various more or less idiosyncratic assumptions, attempt to derive and supply estimates of the spatial-temporal population dynamics of the broad and fuzzy region referred to as the "Helladic Space"; or more specifically (albeit still spatially and temporally fuzzy) of those spatial settings referred to as: Mainland Greece, Egypt, Mesopotamia, Apennine Peninsula, Asia Minor, Balkans, and the Levant.

If one consults the source in ref. [NR.1.1], then the following very rough current Lower and Upper estimates of the World population are obtained for two specific points in time: as of 1950 those bounds stood (in millions) at 2400 and 2560 correspondingly; and as of 200 BC they were estimated to range between 150 and 230 correspondingly; then from the above-mentioned settings' population estimates one concludes that the then population stock globally was at the Lower bound about 6.25%, and at the upper bound close to 9% of the 1950's population stock.

The author takes exceptions to these estimates, in so far as the BC numbers are concerned, for a host of reasons. If such were the population sizes of these settings, there would have been far more spatially extended (albeit still in ruins) remains of their settlements. Admittedly, many of the housing and other residential and non-residential structures were made from non-durable construction materials. But their durable structures would have been of a different (far greater) scale than the one we routinely come across from archeological excavations of sites belonging to that time period.

A closer to their (variable always) size at the period under review would be half to three quarters of what the above BC numbers seem to indicate. In any way, and for exposition purposes (since this is not the primary focus of this paper to analyze) the above numbers will be adopted and processed here.

The Helladic Space's total population share of that Era's World population is also unknown. Only very gross approximations to that count exist, with questions easily raised as to how reliable these estimates can be; for such an estimate, see ref. [NR.1.2]. There, it is mentioned that, the level of Greece **plus** Asia Minor population stock sizes at c. 500, 400 BC and 1 AD fluctuated and stood at around 10, 20, and 16 million correspondingly. Again, the author's cautionary previous *caveat* applies.

If one were to take an approximate average of about 15 million as the population of the Helladic Space during the time period under review (i.e., c. 700 BC – c. 300 AD), then that relative population at around 200 BC represented either about a tenth (at the Lower end) or 6.5% (at the Upper end) of the then World's aggregate population stock size. The reader is again advised of course to keep in mind the applicable caveats.

Then, the 55 mathematicians and astronomers on the List, belonging to a geographical setting that comprised about ten percent of the global population during a century in which the number of distinguished mathematicians reached twelve, represent (using the Theory of Proportions, and assuming that the share of notable mathematicians and astronomers then is the same as contemporary equivalent shares) stands for our 21st century with a global population as of the writing of this paper of around 8 billion, see ref. [NR.1.3], at about: Greek Helladic Space population equivalent 800 million (at the Lower end) and 520 million (at the Upper end), with 9600 distinguished mathematicians (at the Lower end) and 6240 (at the Upper end).

If an Archimedes type mathematician is one out of ten great mathematicians, as suggested by the List, then these ratios imply 960 and 620 Archimedes' equivalents correspondingly (presuming that Archimedes is one of the six individuals mentioned in the List and classified as being a part of the top in the four-tier hierarchical breakdown scheme of that List. This gives an approximate idea of the infrastructure's scale involved in the Ancient Greek Mathematics world of the millennium examined here. It also points to the inherent limitations of the population estimates supplied by the references used (in absence of any other, more reasonable, estimates).

In concluding this Note, the author notes that by considering the glaring impossibility embedded in the above derived conclusion, one by a feedback process could moderate and alter to a degree the BC estimates on the Helladic Space's population to which this author expressed reservations. All that however, is left to future research.

Note 2. On the Greek Dark Age

Of course, as stated in numerous ways and at various points in the main text, this topic having to do with what is classified as "Greek Dark Age" is a huge topic to cover, and far greater than any short Note's treatment of it. Hence, only a few points, hopefully of central import, will be made.

What is striking about this approximately three-century long period is that the Helladic Space emerged very differently in many respects than how it went into that temporal segment. A profound transformation apparently took place, which as it plunged the area (and the region) into a chaotic state, it also transitioned it to a new environment that eventually produced a Civilization of magnificence. And Mathematics *cum* Astronomy played a pivotal role in that transformation at the aftermath of that transition. Much has been written and said, about the period that followed the collapse of the Minoan and Mycenaean civilizations. Very few of these theories, propositions, assertions, hypotheses have full-proof and beyond reasonable doubt standing however. And for good reason: absence of written documentation. Writing's importance in that regards can't be overstated.

Evidence is extremely fragmentary, often inconsistent, largely incomplete, and exhibits striking elements of fuzziness to base any definitive statement about that Era and accept it with any degree of certitude, far more so than the Era that followed. Only grossly speculative holdings on various parts of that period's events can one possess. For an informative, albeit far from being either complete or assertive, in summary type statement, introduction to the Greek Dark Age, see ref. [NR.2.1]. In that reference, one comes across a number of hypotheses, two of which will be reviewed. One of these hypotheses relates directly to **writing**, and it is of central import since writing produces History and rigorous evidence. And in writing, the means employed, i.e., the alphabet used is of extreme import, if not central to the entire issue. The hypothesis (and not only in the above cited references, but frequently encountered in the relevant contemporary literature), is the theory that the Ancient Greeks got their alphabet from the "Phoenicians" (a social group about which very little is known) in the 950 – 750 BC period. It is noteworthy that the Greek language and alphabet (and there are strong links between the two) survived and are both alive to this day, three millennia since their formation and adoption in the Helladic Space. Whereas the "Phoenician" alphabet is not, an interesting event from an evolutionary standpoint.

Of equal import is the evolutionary transition from the Mycenaean Linear B to the Archaic Greek alphabet, a significant event in the field of Helladic Space Linguistics. Thus, some attention will be given to this transition in this Note. In principle, it is asserted that it is next to impossible to prove the statement alluded to from ref. [NR.2.1], about the giving (Phoenicians) and (Greeks) receiving an alphabet on numerous grounds. These grounds will not be addressed here at any length or in any depth, as this is not an issue of central concern to this paper. However, two very brief and cursory points will be made to counter this argument.

First, it is *a priori* extremely difficult to point at an interaction that apparently took place three millennia ago and determine who was on the sending and the receiving end of an **influence**.

Along the same lines, and most importantly, the precise extent of that influence is difficult to gauge. Maybe parts of what is referred to as “Phoenician alphabet” was adopted by the Archaic Greeks; but they certainly transformed it significantly by (among other things) adding vowels to it, see ref. [NR.2.4]. **Second**, what is known and documented by archeological evidence is that the Etruscans (although they also had exposure to the “Phoenicians” possibly as much as the Greeks had, see Figure 4) adopted the Greek writing, i.e., their alphabet, as well as Archaic and Classical Greek cultural elements, see ref. [NR.2.2]. The Etruscans lived in the 900 BC Era, see ref. [NR.2.3]. The Greek type of alphabet that the Etruscans adopted came from the island of Euboea in the Aegean Sea (an island almost attached to the Greek Mainland and only separated by an isthmus). On the Euboean writing, see Figure N2.1. On the Etruscan writing, see Figure N2.2. On the “Phoenician” language albeit in Aramaic writing style, see Figure N2.3. Finally, on a “Phoenician” writing style from the circa 9th to the 8th century BC see Figure N2.4. Comparative analysis of these four Figures containing four different types of writings, is rather easy and quite apparent; the conclusion, is unavoidably fuzzy to an extent but rather clear in rough outlines. Only two of these inscriptions bear rather similar writing styles, and these are the Greek and the Etruscan inscriptions of Figures N2.1 and N2.2.

The Greek Dark Age, remains a very obscure, from History’s standpoint, period for the Eastern Mediterranean Basin. What transpired during the c. 1100 – 800 BC period was of great importance for that Region, but it is still shrouded in deep mystery. The introduction of the Greek Archaic writing (alphabet) was undoubtedly a factor. What magnitude of a role that played in that critical transition that this paper kept referring to remains open for analysis and debate.

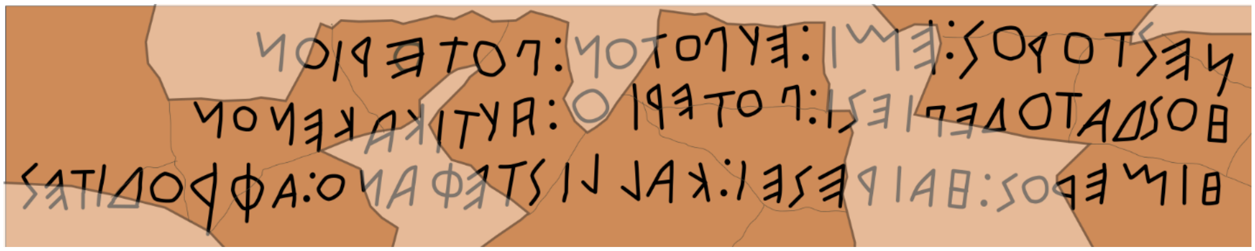

Figure N2.1. Early Greek inscription. The (restored) inscription of the so-called “Cup of Nestor”, found in Ischia, contains the Euboean type alphabet (written right to left in all three lines); it is dated to late 8th century BC. The figure is in the public domain. It was copied from ref. [F.N2.1].

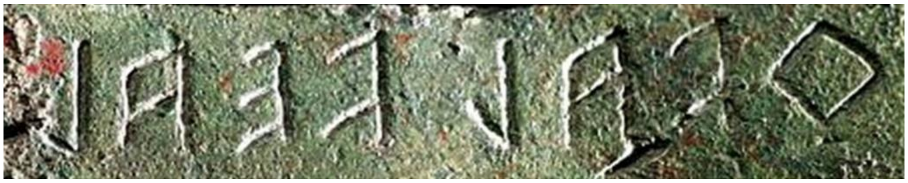

Figure N2.2. Etruscan inscription. This circa 550 BC bronze foil inscription reads: “VEAL THVAL”, and (as translated) it is dedicated to deity Demeter (ΔΗΜΗΤΡΑ), see ref. [NR.2.5]. Source of figure and credit to: ref. [F.N2.2].

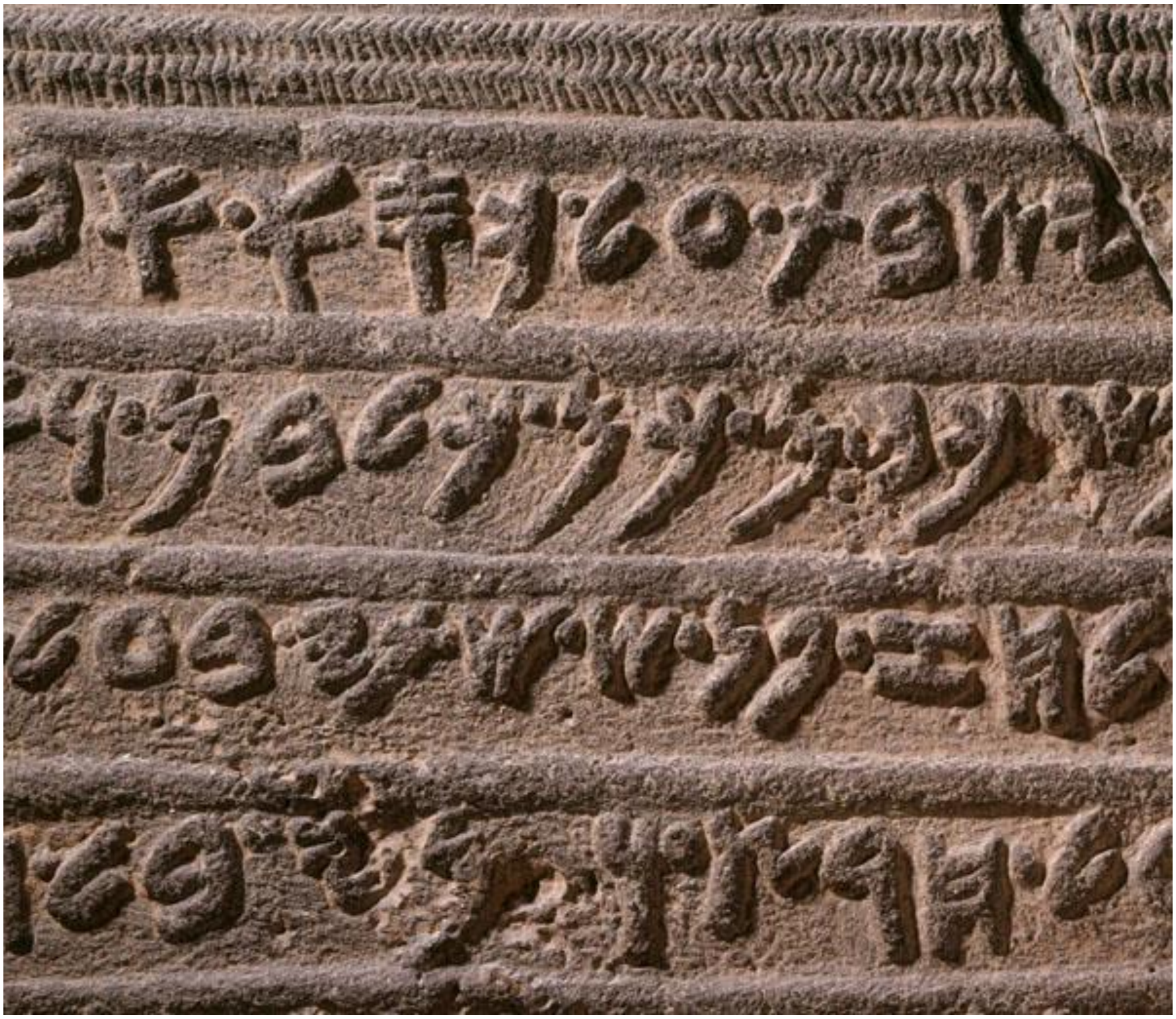

Figure N2.3. Sample from a circa 9th century BC Phoenician language with Aramaic type writing. This inscription (KAI 24) is part of the so-called "Kilamuwa stele", see ref. [NR.2.6], which is also the source of the figure, found in ref. [F.N2.3].

The advent of writing, and the transformation of Minoan Linear A and Mycenaean linear B to Archaic Greek was undoubtedly instrumental, maybe a necessary but certainly not a sufficient condition for this extraordinary transformation. What is shown on Figure 4 of the main text provides a framework for speculative analysis and reflection.

Parenthetically, it must be noted that **fragmentary informal History** was in place in the region surrounding the Helladic Space prior to the Greek Archaic period; the Hittites, the Egyptians and dominions within the broader Mesopotamian region and the Levant (among numerous other settings) had in place language and a writing system. However, that pre-formal History record is too fragmentary and carries with it significant uncertainties to be considered of any value in analyzing and studying influences either to or from these settings form or to the Helladic Space.

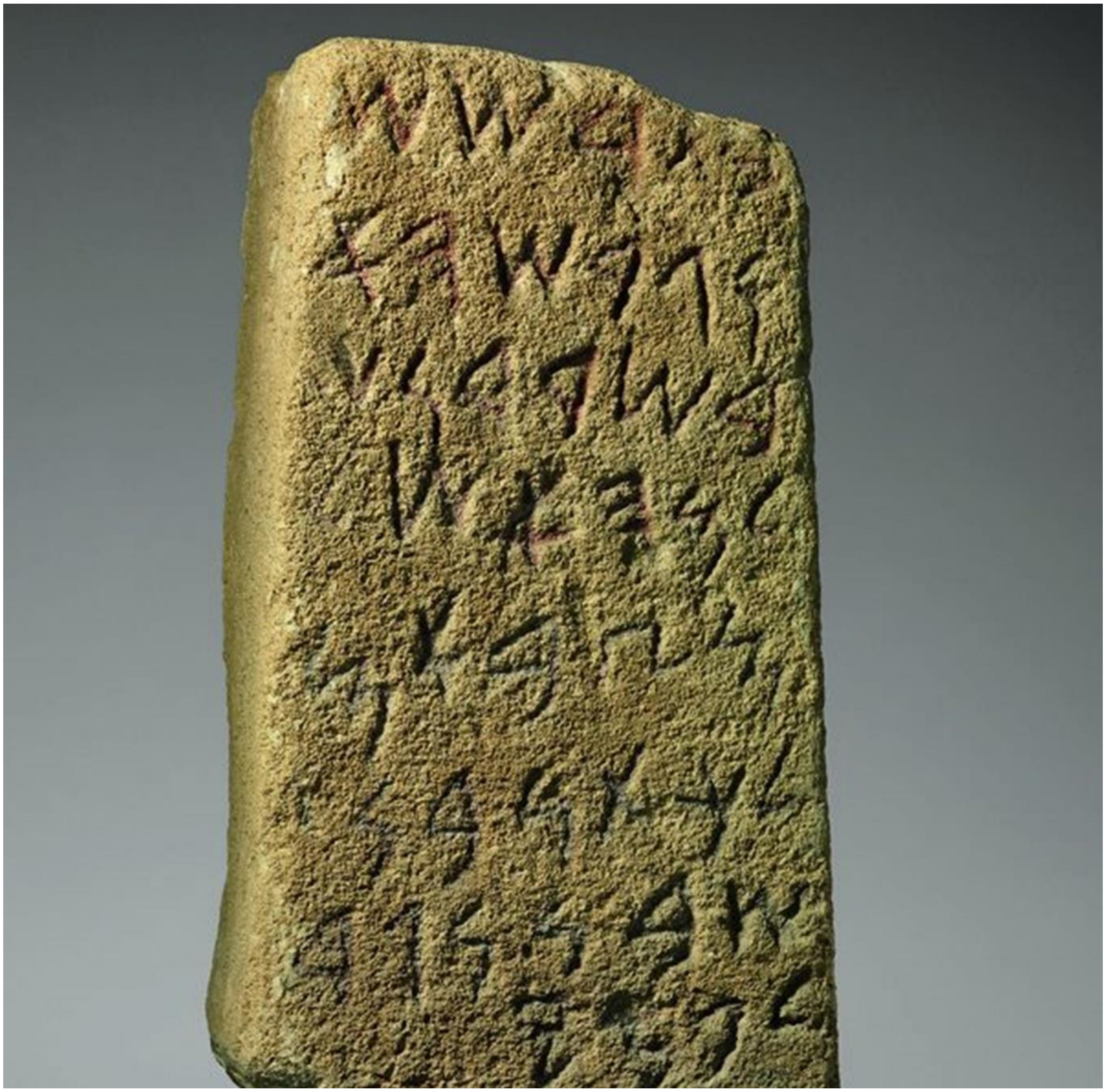

Figure N2.4. Phoenician writing circa 850 – 740 BC. The stele was found near Nora on the island of Sardinia. What is remarkable with regards to this stele is the stark differences in the writing style from that of the Greek, Etruscan and Aramaic writing styles shown in Figures N2.1, N2.2 and N2.3. The text remains only in part translated and imperfectly understood. Source of the figure in ref. [F.N2.4].

Multiple (in the thousands) "Phoenician" inscriptions have been uncovered. A good sample is offered in ref. [NR.2.7]. None of them show any reference to Mathematics. Thus, they were highly unlikely to have had any effect on either Greek Mathematics or Astronomy. If one assumes that Mathematics were the foundations for social progress, a thesis of this paper as well as those of ref. [A1] and [A2], then one must conclude that the "Phoenicians" had little to do with either the Greek Dark Age, or the transformation that ended it and brough about Archaic Greece.

The issue remains: who were these “Phoenicians”? Where did they come from? What were they calling themselves, as the Greek name “Phoenician” was bestowed upon them by the Greek writers of that time period? What was their role in the colonization of the Mediterranean Basin? It is well established that they were involved in trade. But was that the only extent of their role? Obviously, a cursory view of the map in Figure 4 raises the issue: were they in competition with the Greeks in colonizing the Mediterranean shores? Was that competition confined to business, or was it broader and more complex than that? What was the connection, if any, between these peoples that Egyptian references call the “Sea People” and the “Phoenicians”? Who were these “Sea People” that seemed to have been active, and aggressively so, contemporaneously with the Greek Dark Age? It is a period of Western European History that remains obscure. But whatever it was, it had a lot to do with the Greek Dark Age, and most importantly with the transition to the Archaic period that this author is interested in exploring further, in follow-up papers.

It is commonly and often casually mentioned that “the Greeks got their alphabet from the Phoenicians”, as for example does ref. [NR.2.1]. However, as to some extent demonstrated by comparing the writing styles (let alone the embedded Linguistics, as well as the complex interactions between alphabets and Linguistics, grammar, etc.) of these Figures’ epigraphic writing, it becomes a formidable task to fully support such assertion. Moreover, as the discussion about “doxography” in the main text pointed out, one should not always overly rely on historians’ reports, as these reports carry the historians’ own biases and limited viewpoints and perceptions.

The Greek language and alphabet have shown extraordinary durability and resilience, both being still alive over a period of about three millennia. Both have been subjected, of course, to considerable and multifaceted transformations over this lengthy and turbulent time period, experienced over a multiplicity of spatial contexts. They have influenced the alphabets and languages of other cultures, such as Latin and numerous modern languages (including English).

But what is relatively clear from a brief review of the Archaic Greek language, alphabet and writing styles is that there were multiple variants of all three even within the Helladic Space, such as the Attic, Doric, etc. Local **dialects** from the Archaic period which affords us a written record clearly shows that there has been an extraordinary variety of all three. That variety, and competition for linguistic dominance may have been one of the contributing factors to the fiercely competitive (and often extraordinarily hostile) behavior that History has to a large extent recorded to have existed among Greek City States. The spatial-temporal variety along all three dimensions (language, alphabet, writing style) within the Helladic Space, over the study period, seem to indicate that it rather improbable that all of these settings, in unison and synchronicity, adopted the “Phoenician” alphabet in concert sometime in the 8th century BC.

A second hypothesis which is postulated in ref. [NR.2.1] is that rampant poverty was prevailing in the Mainland and Aegean part of the Helladic Space during the Greek Dark Age. It is argued that the social stratification, which prevailed during the Mycenaean period, accompanied the collapse of citadels, their impressive megarons (ΜΕΓΑΡΑ) and cyclopean walls.

It was apparently succeeded (according to the author of ref. [NR.2.1]) by a period of **egalitarianism** during that critical temporal frame referred to as the Greek Dark Age. Moreover, always according to the same reference, by the end of that obscure period, archeological excavations seem to point to a revival of a hierarchical socio-economic stratification, with an economically affluent class reappearing *post* the Mycenaean Era. This hypothesis is mostly pegged to evidence from burials, where high concentration of various luxury items, indicative of relatively high concentration of wealth to a few, apparent chieftains and their immediate families, were found in the circa 950 BC time frame.

Hence, one hypothesis that could account for the socio-economic conditions prevailing at that Dark Age period is that possibly a mass social upheaval or revolt (of unknown spatial-temporal strength and duration) might have been one of the factors in (and likely the major cause of) what plunged the Helladic Space into that Dark Age. Nonetheless, the possibly explosive endogenous population growth hypothesis; ecological misfortunes; along with a possibly violent in-migration (invasions from the North) of Doric tribes into the Helladic Space during the Iron Age, are all factors that should also not be underestimated. Obviously, the Greek Dark Age is (and will likely remain) an historical quantum, where currently co-existing hypotheses about possibly multiple causes and effects will still interact, without any clear resolution in sight. Moreover, it must be noted that this obscure Era’s socio-economic structuring was relatively **short-lived**, lasting about three or so centuries; and that the following Era (Archaic Greece) manifestly had a relatively long-term large in scale effect. The aftermath of that possibly violent social upheaval (revolution) was the eventual restoration of kingdoms and social stratification. A cursory view of mathematicians and astronomers’ social status (those mentioned here), as well as in the broader study of ref. [A1], seems to indicate that almost all of them enjoyed a relatively high socio-economic standing. But it should also be mentioned that none of these individuals’ tombs are either known or extant.

At the end of the Greek Dark Age, one comes across influences from the East (the Levant), as well as from the South (Egypt) in Greek Architecture and Art. In Architecture, the Doric columns of Greek Archaic temples have a precedent in Egypt, see ref. [A11]; from the East, the Ionian column had a precedent in Persia. The Archaic Kouros was heavily influenced by the Egyptian statues. Of course, these influences accompanied the largely unrecorded impacts of the Doric invasions from the North and any possible cultural influences from the Balkans and the West. It must be presumed (not proven) the Helladic Space was at the crossroads of influences from all directions.

The Homeric orally transmitted Epics were sung during the Greek Dark Age before transitioning to their written form during Archaic Greece. They refer to the Mycenaean world at the tail end of the Iron Age, without any reference to the medium that transmitted and channeled them, the Greek Dark Age. This is a peculiar condition, in need of some explanation. It is another mystery in search of an answer, or likely in search of multiple answers, but some answers nonetheless. It is highly unlikely and quite doubtful that such answers will be found in the fragmentary written record of pre-formal History and in spatial settings surrounding the Helladic Space. Inscriptions on stelae or stone, clay tablets or papyri are not likely to reveal any conclusive clues on this count.

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Dedication

This paper is dedicated to the memory of five remarkable and extraordinarily talented modern day Greek personalities: two great music composers, ΜΑΝΟΣ ΧΑΤΖΙΔΑΚΗΣ (1925 – 1994) and ΜΙΚΗΣ ΘΕΟΔΩΡΑΚΗΣ (1925 – 2021); two Nobel Laureates in Literature ΟΔΥΣΣΕΑΣ ΕΛΥΤΗΣ and ΓΕΩΡΓΟΣ ΣΕΦΕΡΗΣ; and to the great troubadour ΓΡΗΓΟΡΗΣ ΜΠΙΘΙΚΩΤΣΗΣ. Cultural icons par excellence, their extraordinary music, literature and voice channel, draw and herald links from contemporary Greece to the Era of Byzantium, down and back to the Era of Ancient Greece and the Helladic Space. All of them significantly contributed to the educational and intellectual experiences of this author during the 1960s, while he was a student of Architecture at the Aristotle University of Thessaloniki, in Greece.